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ABSTRACT

Seven papers presented at a research conference on space and geometry are contained in this monograph. The first paper gives an historical sketch of the development of geometry and discusses several considerations for selecting geometric content for the elementary school. Two papers deal with Piaget's research into the child's development of space and geometry concepts, and another paper suggests directions for further research on space from the Piagetian perspective. A fifth paper reviews the van Hiele levels of development in geometry and discusses the new Soviet geometry curriculum, another paper reviews cross-cultural research on perception, and the final paper examines some research issues concerning children's concepts of transformation geometry. (DT)

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SE 021 659

SPACE

AND

GEOMETRY

Papers from a Research Workshop

Sponsored by The Georgia Center
for the Study of Learning and
Teaching Mathematics
and the
Department of Mathematics Education
University of Georgia
Athens, Georgia

J. Larry Martin, Editor
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and Environmental Education
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MATHEMATICS EDUCATION REPORTS

The Mathematics Education Reports series makes available recent analyses and syntheses of research and development efforts in mathematics education. We are pleased to make available as part of this series the papers from the Workshop on Number and Measurement Concepts sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics.

Other Mathematics Education Reports make available information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific subpopulations of the mathematics education community.

Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the ERIC/SMEAC Center.

Jon L. Higgins
Associate Director

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Acknowledgements and Overview

The Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) was started July 1, 1975, through a founding grant from the National Science Foundation. Various activities preceded the founding of the GCSLTM. The most significant was a conference held at Columbia University in October of 1970 on Piagetian Cognitive-Development and Mathematical Education. This conference was directed by the late Myron F. Rosskopf and jointly sponsored by the National Council of Teachers of Mathematics and the Department of Mathematical Education, Teachers College, Columbia University with a grant from the National Science Foundation. Following the October 1970 Conference, Professor Rosskopf spent the winter and spring quarters of 1971 as a visiting professor of Mathematics Education at the University of Georgia. During these two quarters, the editorial work was accomplished on the proceedings of the October conference and a Letter of Intent was filed in February of 1971 with the National Science Foundation to create a Center for Mathematical Education Research and Innovation. Professor Rosskopf's illness and untimely death made it impossible for him to develop the ideas contained in that Letter.

After much discussion among faculty in the Department of Mathematics Education at the University of Georgia, it was clear that a center devoted to the study of mathematics education ought to attack a broader range of problems than was stated in the Letter of Intent. As a result of these discussions, three areas of study were identified as being of primary interest in the initial year of the Georgia Center for the Study of Learning and Teaching Mathematics--Teaching Strategies, Concept Development, and Problem Solving. Thomas J. Cooney assumed directorship of the Teaching Strategies Project, Leslie P. Steffe the Concept Development Project, and Larry L. Hatfield the Problem Solving Project.

The GCSLTM is intended to be a long-term operation with the broad goal of improving mathematics education in elementary and secondary schools. To be effective, it was felt that the Center would have to include mathematics educators with interests commensurate with those of the project areas. Alternative organizational patterns were available--resident scholars, institutional consortia, or individual consortia. The latter organizational pattern was chosen because it was felt maximum participation would be then possible. In order to operationalize a concept of a consortia of individuals, five research workshops were held during the spring of 1975 at the University of Georgia. These workshops were (ordered by dates held) Teaching Strategies, Number and Measurement Concepts, Space and Geometry Concepts, Models for Learning Mathematics, and Problem Solving. Papers were commissioned for each workshop. It was necessary to commission papers for two reasons. First, current analyses and syntheses of the knowledge in the particular areas chosen for investigation were needed. Second, catalysts for further research and development activities were needed--major problems had to be identified in the project areas on which work was needed.

Twelve working groups have emerged from these workshops, three in Teaching Strategies, five in Concept Development, and four in Problem Solving. The three working groups in Teaching Strategies are: Differential Effects of Varying Teaching Strategies, John Dossey, Coordinator; Development of Protocol Materials to Depict Moves and Strategies, Kenneth Retzer, Coordinator; and Investigation of Certain Teacher Behavior That May Be Associated with Effective Teaching, Thomas J. Cooney, Coordinator. The five working groups in Concept Development are: Measurement Concepts, Thomas Romberg, Coordinator; Rational Number Concepts, Thomas Kieren, Coordinator; Cardinal and Ordinal Number Concepts, Leslie P. Steffe, Coordinator; Space and Geometry Concepts, Richard Lesh, Coordinator; and Models for Learning Mathematics, William Geeslin, Coordinator. The four working groups in Problem Solving are: Instruction in the Use of Key Organizers (Single Heuristics), Frank Lester, Coordinator; Instruction Organized to use Heuristics in Combinations, Phillip Smith, Coordinator; Instruction in Problem Solving Strategies, Douglas Grouws, Coordinator; and Task Variables for Problem Solving Research, Gerald Kulm, Coordinator. The twelve working groups are working as units somewhat independently of one another. As research and development emerges from working groups, it is envisioned that some working groups will merge naturally.

The publication program of the Center is of central importance to Center activities. Research and development monographs and school monographs will be issued, when appropriate, by each working group. The school monographs will be written in nontechnical language and are to be aimed at teacher educators and school personnel. Reports of single studies may be also published as technical reports.

All of the above plans and aspirations would not be possible if it were not for the existence of professional mathematics educators with the expertise in and commitment to research and development in mathematics education. The professional commitment of mathematics educators to the betterment of mathematics education in the schools has been vastly underestimated. In fact, the basic premise on which the GCSLTM is predicated is that there are a significant number of professional mathematics educators with a great deal of individual commitment to creative scholarship. There is no attempt on the part of the Center to buy this scholarship--only to stimulate it and provide a setting in which it can flourish.

The Center administration wishes to thank the individuals who wrote the excellent papers for the workshops, the participants who made the workshops possible, and the National Science Foundation for supporting financially the first year of Center operation. Various individuals have provided valuable assistance in preparing the papers given at the workshops for publication. Mr. David Bradbard provided technical editorship; Mrs Julie Wetherbee, Mrs. Elizabeth Platt, Mrs. Kay Abney, and Mrs. Cheryl Hirstein, proved to be able typists; and Mr. Robert Petty drafted the figures. Mrs. Julie Wetherbee also provided expertise in the daily operation of the Center during its first year. One can only feel grateful for the existence of such capable and hardworking people.

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Teaching Strategies

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Overview

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Since the 1960's there have been many questions raised and statements made in the professional literature about what geometry should be in the curriculum, why (or if) it should be there, when it should be taught, and how it should be taught. As a result, "more" geometry is now included "earlier" and "informally" or at an "intuitive level." Feelings of uneasiness among mathematics educators remain. Answers, if indeed they can be so called, such as "more," "earlier," and "informally" are inadequate. Alterations of the geometry content in the curriculum have tended to be tentative gropings toward some ideal of educational pragmatism. Pragmatism is not inherently bad. Admittedly there is merit in a curriculum that is both teachable and learnable. However, attempts at developing such a curriculum would be less labyrinthine if they would be made within a theoretical framework that took into account both the nature of the child and the structure of the mathematics involved.

It is not surprising that applications of an underlying theory of the child's conception of space or the child's conception of geometry have been minimal. Existing theory itself is minimal. Some mathematics educators have turned to Jean Piaget's work to provide such a theory. Piaget has carried out a great number of experiments dealing with the child's conception of space and/or geometry. His research is within the broader context of his theory of cognitive development and the nature of knowledge. Yet there is not unanimity among mathematics educators about how his work should be interpreted nor, indeed, even if it is relevant to mathematics education.

The research workshop on space and geometry sponsored by The Georgia Center for the Study of Learning and Teaching Mathematics was intended to stimulate dialogue among mathematics educators with the objectives of synthesizing existing knowledge concerning the child's conception of space and geometry and identifying, coordinating, and generating related studies. The papers contained in this monograph were presented at the workshop and provided the stimulus for what is hopefully only the initial dialogue.

Edith Robinson presents a historical sketch of the development of geometry and demonstrates that there are many alternate approaches for selecting the geometry content for the elementary school. In fact, there are many different geometries from which to choose.

One basis for choosing would be the nature of physical reality; this is, select the geometry which is most nearly isomorphic to the "external world." But Charles Smock warns that such a choice of a mathematical model may be self-fulfilling. He points out that a literal interpretation of Piaget's theory necessitates viewing reality as a black box. The child constructs his universe and then experiences it as though it were external to himself. Thus we could never know what is "real," only what we have constructed as real. It is a startling but intriguing idea. At the very least, it focuses attention on the child rather than treating him as the black box. In his paper Smock provides a summary of much of Piaget's early space and geometry research and describes critical features of Piaget's thinking concerning the child's development of space and geometry concepts.

Izaak Wirszup notes that the Russians have accepted many of Piaget's tenets. However, the work of the van Hieles has inspired Russian research more directly. Professor Wirszup reviews the van Hiele levels of development in geometry and discusses the new Soviet geometry curriculum. The reader will notice obvious similarities between Piaget's theory and the van Hieles' theory. Piaget has stages; the van Hieles have levels. The van Hieles have isolated networks of relations; Piaget has figurative knowledge. Yet there are also notable differences. The van Hiele levels appear to deal more with geometric forms; Piaget deals more with transformations. Piaget provides age guidelines for his stages; are there similar age guidelines for the van Hiele levels? Also in comparison with Piaget's research, both the van Hieles and the subsequent Russian research are oriented more towards "curriculum" and "teaching."

While Smock reviews the early work of Piaget, Jacques Montangero reviews more recent Genevan research. Two experiments by Greco focus on the child's organization of spatial representations. One study utilizes Euclidean transformations and the other transformations on a Moebius strip. Two experiments by Vinh Bang deal with the relations between perimeter and area. In addition to reporting these studies, Montangero discriminates between the figurative and operative aspects of knowledge and between logico-mathematical knowledge and physical knowledge, two quite different distinctions. As Montangero points out in his paper, these distinctions have implications for the classroom.

Directions for further research on space are suggested by Montangero and Smock from the perspective of the Geneva group. They suggest that if new research results are to be added to those that exist in space, a change in research method is necessary. In this view, four advantages and four limitations of a structural approach to the study of space are presented. Capitalizing on the limitations, they suggest programs of research, through example, which hold promise for new results. The authors point out emphatically that an "intermediate" body of research is necessary for the results of the Genevans to be applied to educational practice.

Cross-cultural studies related to the child's geometrical and spatial concepts based on Piaget's work are few. In fact, as Michael Mitchelmore points out, there is little cross-cultural research on geometrical concepts per se of any sort. Mitchelmore does provide, however, a thorough review of the rather extensive cross-cultural research on perception. Since there are so many subpopulations in the United States alone, this review should be of special interest. Mitchelmore speculates on the causes of the differences found in different cultures. He also warns that alternative explanations are usually available.

Readers should note that often it is difficult to determine whether a given task is perceptual, in the Piagetian sense (see Smock's paper), or requires spatial representation. Borderlines are not always well-defined. And not all investigators use the word "perception" in the same sense. Thus careful analyses of the studies which Mitchelmore reviews could yield clues about conceptualization as well as perception.

Richard Lesh uses transformation geometry as a context within which to discuss relationships among mathematical structures, cognitive structures, and instructional structures. He examines proposed justifications for including geometry in the elementary school curriculum. But these justifications frequently rely on assumed, albeit unverified, relationships between, for example, mathematical structures and cognitive structures. Lesh suggests research techniques appropriate for investigating the nature of such relationships.

The intent of the preceding paragraphs has been to provide the reader with a brief overview of the papers in the monograph. What follows is a collection of observations, comments, suggestions, and questions generated by the papers and the resulting discussions. Some are reiterations of what is contained in the papers. Other portions may appear only obliquely related to the papers. All are impressions presented here in an attempt to capture the spirit of the workshop. To that end, not all the ideas are developed thoroughly, nor are all questions answered. But hopefully they will help to give the reader a sense of the rich potential for research in space and geometry.

What is the purpose of geometry in the elementary school? Should instruction be aimed at developing the child's concept of space or the child's concept of geometry? My opinion is that the instruction should be aimed at assisting the child develop a well-organized concept of space. This does not mean that there would not be many geometrical concepts in the curriculum But the concepts would be those necessary for achieving the primary objective.

Mathematics educators must distinguish between perceptual, representational, and conceptual space. Once we have made these distinctions, we need to act like we have made them. Recognizing that at the pre-operational level perceptions may dominate conceptions, we must also allow that perceptions do not stand alone. Conceptions influence perceptions. Currently the curriculum contains mostly perceptual tasks. Honest efforts to study the child's representational and conceptual space are necessary.

It is also imperative to distinguish between the figurative and operative aspects of knowing. It appeared to me that the term "figurative thought" was often used with a somewhat negative connotation at the workshop. Such connotation is not inherent to figurative thought. While Piaget emphasizes operative thought, he recognizes that figurative thought is especially important when space conception is involved. Figurations can serve as an aid to operations. Yet little is known about the role that figurative thought plays in the child's construction of space.

Invariance through transformation should be emphasized in research and in the curriculum. This does not necessarily mean emphasizing the transformation itself. As Lesh points out, focusing attention on the transformation may only serve to confuse the child. Do children think in terms of transformations? Does he think in terms of results (end points) or does he actually consider how he might get from one point to another?

It is obvious that mathematical and psychological uses of the same terms do not always coincide. Much more needs to be known about the relationship between cognitive "structures" and mathematical "structures" and how one may assist in developing the other. Do physical "transformations" or mathematical "transformations" effect mental "transformations"? When Piaget speaks of topological concepts what does he mean? Studies are needed which analyze from a mathematical point of view the mathematics involved in Piaget's tasks, for there is much mathematics there.

The Erlanger Programm has been appealed to as a model--a model for what? The child's construction (process) of space? For research? In what sense is it a model? It definitely can be used to formulate research questions. The Programm speaks of a set X, a group of transformations and invariants. Research need not be restricted to a particular group of transformations nor even to a specific set X. The Programm does a nice job of organizing transformations and displaying the invariants. By analyzing the resulting structures many researchable questions arise (see Martin, 1976) about ideal points, sequence of development, neighborhoods, continuous functions, etc.

Mathematics educators must ask our own questions. We must not expect psychologists, for example, to ask the questions of importance to us, let alone answer them. But this does not mean we should ignore their findings. Lines of communication must be kept open, indeed strengthened. Piagetian theory, if it cannot be accepted in toto, need not be rejected in toto. Can it be adapted and expanded upon to fit our needs? How do the Soviet studies come in? Most of us cannot answer this last question because we haven't read them. We need to. Analytical comparisons between Piaget's theory and van Hiele's theory could prove fruitful.[1]

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[1] David Cilley at Northwestern University is currently studying the van Hiele levels and is trying to find ages for each level.

The frequency of examples, illustrations, anecdotes, and analogies drawn from the real number system to make a point was striking during the first day of the conference. The number of such illustrations drawn from space and geometry was comparatively small. This was dismaying at a conference whose main concern was space and geometry. But it demonstrates how little we know about the child's construction of space.

References

Martin, J. L. The Erlanger Programm as a model for the child's conception of space. In A. R. Osborne & D. A. Bradbard (Eds.), Models for learning mathematics. Columbus, Ohio: ERIC/SMEAC Science, Mathematics, and Environmental Education Information Analysis, 1976.

Mathematical Foundations of the Development of Spatial and Geometrical Concepts

Edith Robinson

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In considering the mathematical basis for elementary school geometry, we are faced with the difficulty that there is no consensus, either in theory or practice, as to what geometry should be taught in the elementary school. Although the inclusion of more geometry has been advocated for decades, implementation has taken a variety of directions. The curriculum has been augmented by such diverse alternatives as additional vocabulary (new and old), modified content from high school (Euclidean) geometry, puzzles of antiquity, and new games and hardware. Moreover, grade placement of topics has shown comparable diversity: The study of area, for example, has been introduced as early as grade 3. To some extent, the current situation reflects our lack of knowledge as to what geometry elementary school children can learn, but it seems unlikely that this will soon be remedied unless there is a strong commitment about both the amount and kind of geometry that elementary school children should learn.

Since the matter is as yet unresolved, this paper will be separated into two parts. In the first part, the history and current status of geometry will be discussed; in the second part, implications for the elementary school will be considered.

Historical Development of Geometry

There seems little doubt that as early as 2000 B.C., some geometry was known to Babylonians and Egyptians. They were familiar with means of computing areas of rectangles, right triangles, isosceles triangles, and possibly the general triangle. What is now known as the Pythagorean Theorem was also known. Some of the accepted facts were incorrect. For example, the area of the general quadrilateral was taken to be $\frac{1}{4}(a + c)(b + d)$ where a, b, c, and d are lengths of consecutive sides. On the other hand, if an obscure passage in Herodotus is interpreted to mean that the area of each triangular face of the great Pyramid (erected c. 2900 B.C.) is the square of the vertical height, a relationship closely supported by present-day measurement, then the builders may well have been familiar with the Golden Section. Herodotus, together with the Rhind and Moscow papyri, furnish considerable information about the procedures devised for computing areas and volumes.

Thales (c. 640-550 B.C.) is frequently credited with perceiving the deductive possibilities in geometry. A wealthy merchant, he made numerous trips to Egypt, and upon his retirement at an early age, took up the study of philosophy and mathematics. During his visits to Egypt, he had become acquainted with geometry and had calculated the height of the Great Pyramid from the length of its shadow. He also established that an angle inscribed in a semicircle is a right angle, that the base angles of an isosceles triangle are the same size, and he is believed to be the first to recognize the importance of studying loci. In his later years, he advised one of his pupils, Pythagoras by name, to go to Egypt to study mathematics. From the Pythagorean school came much of the geometry that later appeared in Euclid's Elements. According to Proclus, writing in the fifth century A.D., it was Hippocrates of Chios, a Pythagorean, who attempted the first logical organization of geometry. Somewhat better attempts were made later by others of the Pythagorean school. All in all, these efforts spanned some three hundred years. The overall plan was to develop a "sequence of statements obtained by deductive reasoning from a set of initial statements assumed at the outset of the discourse" (Eves, 1963, p. 12). About 300 B.C., Euclid, the first professor of mathematics at the famed University at Alexandria, collected, organized, and supplemented known results in geometry into the thirteen books of the Elements. The Elements was but one of the several books by Euclid. There was one on geometric fallacies, one on spherical geometry, one on surface loci, one on optics (which treated perspective), one on conics, and one with the mysterious title, Porisms. All of these are mentioned by Proclus, but all have been lost except the Elements. The descriptions, however, testify to the fact that by Euclid's time, a considerable amount of geometry had been explored.

The days of the Roman Empire and the Dark Ages saw little new activity in geometry. During this time, there were several translations of the Elements into Latin: some of these from the Arabic, some from the Greek. The 1572 translation from the Greek became the source of the English translation.

The revival of learning, however, brought with it a new interest in Euclid's famous fifth postulate. From earliest times this had been thought to differ in character from the other postulates.[1] Writing in the fifth century A.D., for example, Proclus, after stating the postulate, says, "This ought to be struck from the postulates altogether. For it is a theorem" (Proclus, 1970, p. 150).

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[1] Euclid distinguished between axioms and postulates. No such distinction will be made here: the two words will be used interchangebly.

This view had been espoused by many over the centuries, but attempts to prove this "theorem" always resulted in the discovery of some flaw in the argument. Finally in 1733, the Jesuit priest Girolamo Saccheri, Professor of Mathematics at the University of Pavia, and an accomplished logician, published a book entitled Euclides ab omni naevo vindicatus (Euclid Freed of Every Flaw). In this, he attempted, by the method of reductio ad absurdum, to prove the parallel postulate. The sequence of theorems he developed for this purpose include many which have now become classics of so-called non-Euclidean geometry. The "contradiction" which he finally obtained, however, was based on a vague observation that lines just don't behave that way. The diagram used by Saccheri is shown in Figure 1. In this diagram, angles A and D are right angles, and sides $\overline{AB}$ and $\overline{CD}$ are the same length. He was able to establish that angles B and C were the same size, and that they were not both obtuse. He was unable to prove, however, without the parallel postulate, that they could not both be acute, except by the weak argument cited above. Later it was shown that Euclid's parallel postulate was equivalent to postulating that a rectangle exists--it will be noted that Saccheri was unable to prove satisfactorily that the quadrilateral shown was a rectangle.

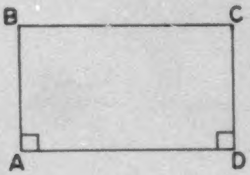


Figure 1

Saccheri's book was withdrawn from the market relatively early. During the eighteenth century, two other mathematicians, Lambert and Legendre, also made attempts to prove the fifth postulate by the method of reductio ad absurdum. Both encountered the same difficulty as Saccheri, namely that of disproving the acute angle case.

The fifth postulate, as stated by Euclid is:

> If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continuously produced, shall at length meet on the side on which are the angles which are less than two right angles. (Todhunter, 1955, p. 6)

The version which is more familiar is the Playfair axiom:

> Through a given point A not on a given line m there passes at most one line which does not interset m. (Eves, 1973, p. 445)

Between 1792 and 1813, the Prince of Mathematicians, Gauss, attempted a proof of the Playfair axiom. Failing in all of his attempts, he developed a geometry in which this did not hold, but apparently fearing ridicule from the mathematical community, never published his results. Upon his death in 1855, however, his interest in the problem was discovered, and attention thus drawn to two other papers. One of these, by the Russian Nicolai Ivanovitch Lobachevsky, had been published in 1829, and the other, by the Hungarian János Bolyai, had appeared in 1832. Both Lobachevsky and Bolyai had independently developed geometries in which it was taken as a postulate that through a given external point, at least two lines can be drawn parallel to a given line. One consequence of this axiom is that the sum of the measures of the interior angles of a triangle is less than two right angles, whereas by Euclid's postulate, the angle sum is exactly two right angles. Which is true? Reputedly both Gauss and Lobachevsky attempted to settle the matter "in the large"--Gauss by taking a triangle formed by three mountain peaks, and Lobachevsky with astronomical distances. Both got results that differed from 180°, but by no more than could be accounted for by experimental error. Thus the matter was not settled.

The furor caused by the discovery of non-Euclidean geometry resulted in close scrutiny of the foundations of mathematics. The whole question of formal axiomatics was explored, with emphasis on consistency--that is, freedom from contradiction. In 1868, Beltrami exhibited a consistent model for Lobachevsky-Bolyai geometry within Euclidean space, thus showing it to be consistent if Euclidean geometry is. The new interest in foundations gave rise to several new sets of postulates for Euclidean geometry which remedied the various logical flaws. (For example, using Euclid's postulates, it is possible to prove that all triangles are isosceles.) Of the dozen or so such sets of postulates, those of Hilbert (1899, subsequently revised) and Birkhoff (1932) are probably best known. As an indication of the continuing interest in this aspect of geometry, another set of postulates for Euclidean geometry was developed by Levi in 1960.

In 1854, Riemann showed that another consistent geometry could be based on the assumption that any two lines in a plane meet; that is, through a given external point, no lines can be drawn parallel to a given line. With this development, there were now three geometries based on three different parallel postulates, and a choice of:

1. The angle sum for the triangle is greater than 180°.

2. The angle sum of the triangle is equal to 180°.

3. The angle sum for the triangle is less than 180°.

All have useful models. As to which truly describes physical reality, we simply do not know. There is some evidence (Tuller, 1967, p. 18) that the binocular vision of normally sighted individuals is best described by the hyperbolic model (Lobachevsky-Bolyai). However, as Poincaré pointed out at the beginning of the twentieth century, physical experimentation, as with mathematics, rests on axioms. For example, it is assumed that light travels in straight lines.

The development of geometry was not restricted to the study of the foundations. In 1639, an engineer named Desargues published a treatise on conic sections in which he used the notion of an "ideal point" and an "ideal line" added to the Euclidean plane. At the time the only attention the book received appears to have been ridicule. For one thing, Desargues' style of writing was, by all accounts, tedious (all printed copies of the original manuscript have been lost), and the terminology he employed baffling. For another, Descartes' book in which he described his analytic geometry had appeared in 1637 and this, together with new results in the calculus, served to direct attention away from geometry. The Napoleonic era, however, saw a lively group of mathematicians at the École Polytechnique in Paris, among whom was Gaspard Monge. To solve certain military problems, Monge developed descriptive geometry--the geometric theory of representing three-space figures in two-space. One of Monge's pupils was Jean-Victor Poncelet. Poncelet was later an officer in the Napoleonic army and was taken prisoner during the retreat from Moscow. During the year he was imprisoned in Saratoff, he survived the rigors of prison by recalling and rearranging all that he had learned of mathematics, and he returned to France with material for "seven manuscript notebooks." One of these, published originally in 1822, was a book on projective geometry. The turn of the century had seen the rediscovery of a manuscript copy of Desargues' work, prepared earlier by one of his students. Also, a work by Pascal (another student of Desargues) was rediscovered in which he credited Desargues with having suggested the methodology for the proof of a theorem. Projective geometry now came into its own. The origins of the ideas of projective geometry date back to the Greeks. Apollonius (?262-200 B.C.), for example, wrote on conic sections including cases in which the cone is oblique; Menelaus (?100 A.D.) established the cross ratio property of a transversal drawn across a pencil of lines; and Pappus (?300 A.D.) established the theorem which still bears his name. During the Renaissance, painters had struggled with the problems of perspective; Dürer, writing in 1525, investigated the problem scientifically (i.e., geometrically). Dürer, as a matter of fact, should probably be given much more credit for his role in the development of projective geometry. One of the more intuitive aspects of early projective geometry was the recognition of invariant properties--the viewer of an object perceives different images from different points of view, yet there are certain similarities which enable him to recognize these as images of the same object. At a later time, these similarities were to be more precisely described as "the set of properties preserved under projective transformations." At first, projective geometry developed as an extension of Euclidean geometry; that is, the parallel postulate of Euclid was included. However, during the era of interest in foundations, it was shown that projective geometry is independent of the parallel postulate. Henceforth it was developed as an abstract geometry based on its own set of axioms. Within this framework, Euclidean geometry became a particular case of projective geometry.

The nineteenth century also saw one other important development in geometry. In 1872, upon his appointment to a professorship at the University of Erlangen, Felix Klein presented his definition of geometry as the study of properties which remain invariant under groups of transformations. The nineteenth century had seen the development of the theory of abstract groups; models (that is, realizations) of groups include the symmetries of regular polygons and polyhedra. The Erlanger Program, as Klein's proposal came to be known, represented another link between algebra and geometry (Descartes had provided an earlier link).

Geometry Today

The historical sketch presented above is just a sketch; many important persons and contributions have been omitted. What was included was background information to illustrate three main directions, or themes, which seem to have predominated:

1. Axiomatics. Modern axiom schema must meet certain criteria: Most important is consistency; in addition, independence, completeness, and categoricalness must be met.

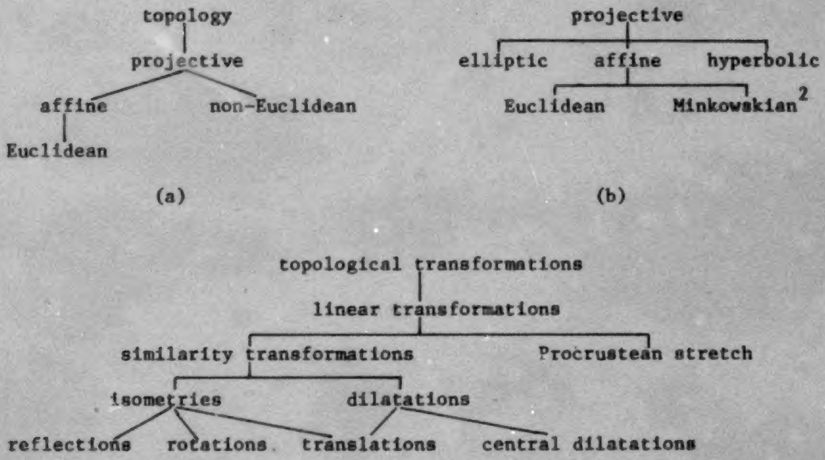
2. Geometries, not geometry. There is, in addition to Euclidean geometry, projective geometry, hyperbolic geometry, elliptic geometry, several Riemannian geometries, inversive geometry, and so forth.

3. Methodology. A gross classification here would be synthetic or algebraic. The former includes sets of axioms with only geometric content; the latter includes "metric" axiom schema, group structures, and vector methods.

These directions, of course, are interdependent. They serve to underscore, however, the flexibility which characterizes "geometry" today. The present day mathematician, attempting to solve a geometric problem, has at his command a variety of methods of attack, and, for a given problem, one of these methods may be substantially easier than another. There are, for example, theorems which are easy to prove with analytic methods but difficult to prove synthetically; there are theorems which are difficult to prove synthetically but easy to prove using groups of transformations; there are also theorems which are easy to prove synthetically but difficult to prove analytically. The present day mathematician may also "borrow" results from one geometry to assist in the solution of a problem in some other kind of geometry; the clarification of axiom systems enables him to determine when this can be done with impunity.

Geometry today may be studied by any one of a variety of approaches. As a matter of fact, the debate that wages today over whether high school geometry should be taught traditionally (a synthetic approach), by vector methods, by transformations, or by some eclectic method (including the coordinate plane), reflects both the availability of multiple approaches and the desirability of the attendent flexibility for problem solving.

Some classifications of geometries. The flexibility alluded to above permits several schemes for classification of geometries. Some of these are diagrammed in Figure 2. Figure 2a is from Meserve (1955, p. vi), and Figure 2b is from Coxeter (1965, p. 19). A scheme for classification which incorporates the Klein definition of geometry is shown in Figure 2c (modified from Coxeter & Greitzer, 1967, p. 101). It should be noted that in Figures 2a and 2b, Euclidean geometry is considered to be a special case of projective geometry.


topology
projective
affine
non-Euclidean
Euclidean
projective
elliptic
affine
hyperbolic
Euclidean
Minkowskian[2]
topological transformations
linear transformations
similarity transformations
Procrustean stretch
isometries
dilatations
reflections
rotations
translations
central dilatations

(a)

(b)

(c)

Figure 2

---

[2] Hermann Minkowski developed a "world geometry" from the general theory of relativity.

To illustrate the first two of these genealogical trees, some axioms for projective geometry are:

P-1: If A and B are distinct points, there is at least one line containing both A and B.

P-2: If A and B are distinct points, there is not more than one line on both A and B.

P-3: If A, B, and C are points not all on the same line, and D and E are distinct points such that B, C, and D are on a line, and C, A, and E are on a line, there is a point F such that A, B, and F, are on a line and also D, E, and F are on a line.[3]

P-4: There exists at least one line.

P-5: There are at least three distinct points on every line.

P-6: Not all points are on the same line. (Meserve, 1955, p. 26)

Since all of these postulates hold for Euclidean and non-Euclidean geometries, any theorem derived from these six axioms holds in either subgeometry. For example, both of the following theorems hold for Euclidean, hyperbolic, or elliptic geometry:

Theorem 1: If two points of a line are on a given plane, then every point of the line is on that plane.

Theorem 2: Any two distinct coplanar lines intersect in a unique point.

On the other hand, because of the differences in the postulates for parallels in the non-Euclidean and Euclidean geometries, we have the following theorems:

1. For Euclidean geometry. Two lines in the same plane which are perpendicular to the same line are parallel.
2. For hyperbolic geometry. Two lines in the same plane perpendicular to the same line are hyperparallel.
3. For elliptic geometry. Any two lines in the same plane intersect.

---

[3] In Euclidean geometry, the point F may be the "point of infinity."

With respect to the classification by transformations, in Figure 2c, there are definitions such as: "A topological transformation is a mapping of the plane onto itself which is 1-1, continuous, and with a continuous inverse" (Gans, 1969, p. 190), and theorems such as:

1. There is a unique similarity transformation that sends a triangle ABC into a similar triangle A'B'C' so that A, B, C goes to A', B', C', respectively. (Gans, 1969, p. 75)

2. Every rotation is the resultant of reflections in two lines through its center. (Gans, 1967, p. 55)

These two types of classification illustrate the flexibility of approach which is possible for the study of geometry today. However, regardless of their seeming dissimilarity, in any formal approach, the geometry is organized into a sophisticated chain consisting of: definition, axiom, theorem, proof. Furthermore, in essence, this situation has obtained since Euclid write the Elements. One must go back to the days of the early Egyptians and Babylonians[4] to find much geometry that is of an empirical nature. This is in contrast to the situation which characterized algebra, in which axiomatic structure came late upon the scene (19th century). Prior to that time, solutions to general classes of problems (e.g., cubic equations) were determined outside of any axiomatic system. Generalization, rather than formalization, characterized developments in algebra. As a consequence, in school we meet the integers first as solutions to equations of the form a + x = b; later they can become a "concrete" model for the algebraic structure known as an integral domain.

Some Curricular Considerations

Implications from the Subject

Obviously, the formal definition-axiom-theorem-proof approach to geometry is not suitable for elementary school children. Also, as just mentioned, the historical development of the subject does not provide the nice curricular model furnished by algebra. Nevertheless, the history and present status of the several geometries do suggest some alternatives for selecting geometric content for the elementary school. (These alternatives are not intended to constitute an exhaustive list!)

---

[4] Some historians believe that much of the same geometry was known to the ancient Chinese and Indians, but that these were not written on materials which could survive the ages.

First, there is the historical route. This has been the traditional one, to the extent that geometry has been taught at all in the elementary school. Mensuration formulae are studied first. Then, in the upper grades, some work is done with similar triangles and indirect measurement. In high school, classical Euclidean geometry is studied; then, in college, other geometries (if any geometry at all) are studied. One problem with this approach is that elementary school children have difficulty with measurement of all kinds, and this serves to delay the introduction of geometry.

Second, there is the route of material axiomatics. In contrast to formal axiomatics, in which objects may have only the properties assigned to them by axiom or definition, material axiomatics assumes the objects of study to have, in some sense, been known prior to the setting forth of the axioms. Essentially, this was the Greek notion as exemplified by Euclid in the Elements. As applied to the elementary school curriculum, the children would be "led" to "discover" such axioms as:

> Given two distinct points, there is one and only one line containing them.
>
> Two lines in the same plane can meet in at most one point.

Some contemporary textbooks seem to take this approach and ask leading questions, the answers to which are essentially the Euclidean postulates.

Third, there is an approach in which children learn certain theorems by experimentation. Activities based on paper folding, for example, seem to illustrate this approach. By folding paper, or by drawing a suitable number of pictures, children can convince themselves that the base angles of an isosceles triangle are the same size. By fitting a rectangular piece of cardboard, or a carpenter's square, into a semicircle, the "truth" of the theorem that an angle inscribed in a semicircle is a right angle can be demonstrated. Tearing triangular sheets of paper substantiates the Euclidean angle sum theorem. Studying equatorial lines on a sphere generates certain theorems of elliptic geometry. Some examples of this kind are to be found in the literature for elementary school mathematics.

Fourth, there is the route suggested by the Erlanger Programm. When considered formally, this route may seem to be impossible. However, the invariants under the various groups of transformations include many of the properties and relations for which definitions are given in any formal course in geometry, viz., parallelism, perpendicularity, congruence for triangles, congruence for angles, congruence for segments, similarity. and betweeness. (Betweeness, of course, may be taken as a "primitive notion" in a synthetic approach, but it is defined in a metric approach.) Transformations themselves can be easily illustrated in an experimental setting. Rotations can be illustrated, for example, by turning a sheet of paper about a point, and projective transformations by casting shadows. Since this approach seems not to have been tried, a few examples may not be out of place here.

The formal definition of a topological transformation given earlier may seem formidable, but it is easily illustrated. If a rubber band is stretched, whether uniformly or not, so long as it is not broken or made to cross itself, that deformation is an example of a "continuous transformation" whose "inverse is also continuous"; that is, it models a topological transformation. Sometimes called "elastic motions," topological transformations can be modeled with kindergarteners' clay, with balloons, and with elastic thread. Two of the invariants are: order along a curve, and interior (exterior) of a simple closed curve ("being a simple closed curve" is also an invariant). Size and shape are not invariants: Circles can be deformed into triangles, into larger (smaller) circles, or into any other kind of simple closed curve. Straightness is also not an invariant under topological transformations; that is, in the same sense that all simple closed curves are "topological equivalent," a line, an angle, and a parabola are also equivalent.

Under projective transformations, straightness is an invariant: The shadow of a straight stick will always be straight. Size and shape, however, are not invariants. The shadow of a stick may be longer than, shorter than, or the same length as the stick, and the shadow of a circle (ellipse) may be an ellipse (circle). Moreover, any topological invariant will be a projective invariant, since projective transformations are particular kinds of topological transformations. The isometries (rotations, reflections, and translations) have size and shape as invariants in addition to all invariants mentioned so far.

It was mentioned earlier that the idea of an invariant property arose in the context of a viewer recognizing an object from different images received as he viewed it from different angles. Apparently, the viewer is attentive to certain visual "cues" in making such judgments. Possibly childen could learn for themselves the cues which tell them whether a particular figure could result from a projection (or rotation or reflection, etc.) of some given figure and, in this way, develop some meaning for terms such as "congruence," "perpendicularity," and "similarity" prior to the introduction of these words. It should be noted that one of the difficulties encountered by students in a course in Euclidean geometry arises with cases of "overlapping triangles," as, for example, triangles ADC and CEA (or triangles ABE and CBD) in Figure 3. In each of these pairs, one is the image of the other under a reflection, and perhaps a transformation approach might alleviate the difficulty.

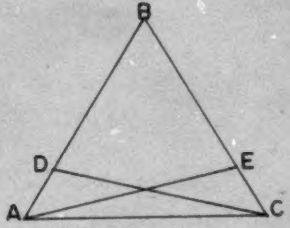


Figure 3

Articulation with High School Geometry

One argument for articulation between elementary and high school geometry is that of preventing compartmentalization of content into unrelated collections of "facts." Such compartmentalization neither reflects the nature of mathematics nor contributes to the mathematical development of the student.

In view of the deductive nature of the usual high school geometry course, articulation may not seem to be possible unless by "articulation" one means a kind of duplication--duplication of the simpler (and easier) parts of the course. As a matter of fact, it has sometimes been recommended that some of the easier parts of the high school geometry course be moved down into the elementary school. However, there are some well-known difficulties with traditional high school geometry, one of which (recognizing overlapping triangles) was mentioned in the previous section. Perhaps articulation could be aimed not so much at preparing students for the easy parts of the course, but for the more difficult.

The example of recognizing overlapping triangles was cited as a possible outcome from studying transformations. The typical method for handling this difficulty in high school is to make it the specific goal for one or more lessons. The lessons might begin with teacher or textbook asserting that there are more than three triangles in Figure 4a, or demonstrating by drawing Figure 4a separated as in Figure 4b. After more examples, the students might be asked to name all the triangles in a figure such as Figure 4c. No outcome for the lessons is expected other than that students be able to recognize instances of overlapping triangles. Furthermore, within this necessarily compact setting, and with only a small number of instances, not all students acquire the requisite proficiency.

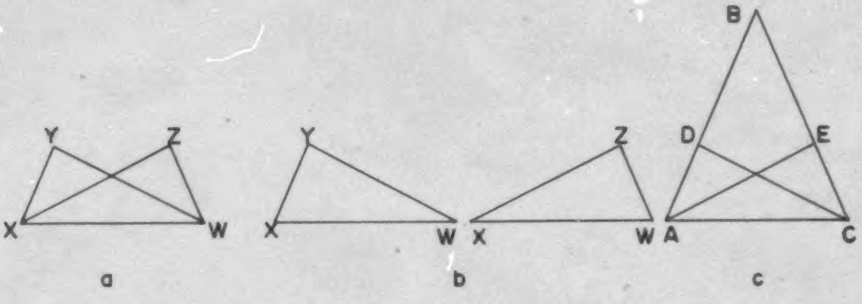


Figure 4

In contrast, if learning to recognize images of figures under transformations is the aim of a series of lessons, and if the pace can be more leisurely, students might not only learn to recognize instances of overlapping triangles (images under reflections) but also to make correct identifications in other slippery cases. In Figure 5, for example, triangle HJK can be the image of triangle KLH, and triangle BMC the image of triangle DMA under (separate) rotations of the plane. In this setting, the student may gain experience with not just one, but three interrelated competencies, lack of any of which can prove troublesome in formal geometry:

1. recognizing pairs of congruent triangles,

2. recognizing pairs of overlapping congruent triangles, and

3. identifying corresponding parts of congruent triangles.

With respect to the third, corresponding parts are those which are images of each other under the transformation, and this, of course, is true when the transformation is a similarity as well as when it is an isometry.

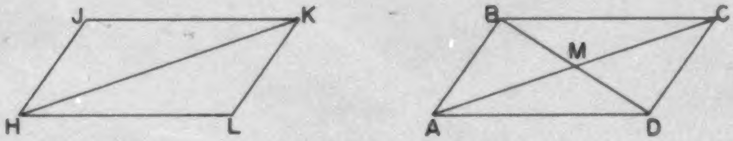


Figure 5

It may be instructive at this point to look at some of the other difficulties and misconceptions of students of high school geometry[5] to see if there are other implications for the elementary school. The most obvious difficulty, of course, is with writing proofs. Another is difficulty in using algebra where needed for solving numerical problems. A third difficulty is with definitions--believing, for example, that any scalene triangle has a hypotenuse. Aside from these, however, there are others which are either more subtle or whose resolution is less clear-cut, but which, nevertheless, impede the student's progress. In the following outline some of these are listed and classified, with no claim that the six categories shown are either mutually exclusive or exhaustive.

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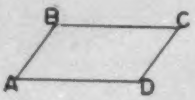
[5] For the remainder of this section, the term "high school geometry" will mean traditional Euclidean geometry taught deductively.

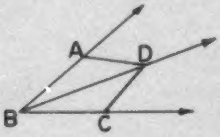
I. There are difficulties with the relational aspects of geometry. Four examples are:

A. Confusion with the meaning of the word "equidistant."

1. $\overline{BC} \parallel \overline{AD} \Rightarrow \overline{AB} \cong \overline{CD}$ because "parallel lines are everywhere equidistant."



2. $\overrightarrow{BD}$ bisects $\angle ABC \Rightarrow \overline{DA} \cong \overline{DC}$ because a "point on the bisector of an angle is equidistant from the sides."

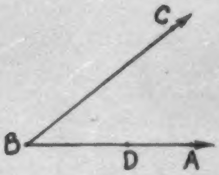


3. For locus problems, "equidistant" is often taken to mean "at a given distance."

B. Confusion with the words "complement" and "complementary." For example, a student may say "A complementary angle equals a right angle."

C. Not understanding proportionality. For example, if the problem is to partition a segment into parts proportional to three given segments, many students do not know what this means, and even when shown the "correct" solution, do not understand why it is correct.

D. Confusion with perpendicularity. For example, when required to construct a perpendicular to $\overline{BC}$ from D, some students draw $\overline{DC}$, and others construct a perpendicular to $\overline{AB}$ at D.



II. Difficulties with measurement. Two examples are:

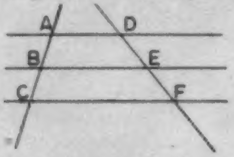
A. Not believing that a square and a triangle can bound regions having the same area.

B. Not believing that the ratios of the areas bounded by similar polygons is that of the squares of the corresponding sides.

III. Drawing unwarranted conclusions[6] from a theorem or from a sequence of theorems. Five examples are:

A. Asserting that when two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are congruent.

B. Asserting that $\overline{AB} \cong \overline{DE}$ because "If a series of parallel lines cut off congruent segments on one transversal, they cut off congruent segments on every transversal."



C. Believing that two triangles are congruent be SSA.

D. Believing that two triangles are congruent by AAA.

E. Believing that any two equilateral triangles are congruent by SSS.

IV. A theorem is not used when needed, or it is used incorrectly. Two examples are:

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[6] These unwarranted conclusions do not appear to be logical difficulties, but rather inability to perceive the "truth" of the situation.

A. The theorem, "The segment joining the midpoints of two sides of a triangle has half the length of the third side" is not used when applicable.

B. When using the theorem, "The side opposite the 30° angle in a 30-60 right triangle has half the length of the hypotenuse," the student takes that side to be one-half the length of the other *leg*.[7]

Before proceeding to the last two categories, it may be noted that the examples so far have to do with the students' belief system, for they are cases of mis-belief or dis-belief. That is, it appears that the mysterious entity called "intuition" is at work. In the case of "complementary angle," the intuition seems to be *lacking*; in the case of the side opposite the 30° angle, the intuition seems to be *faulty*. Perhaps in each of these cases we might say that the student "does not understand the theorem," but that does not tell us very much. Furthermore, if we look at the "interior angles on the same side of the transversal" situation, who could believe that α and β are the same size in Figure 6a? So why would a student say the angles were congruent? One might argue that he does not know the meaning of the phrase, "interior angles on the same side of the transversal," yet Figure 6a does not assist the student in arriving at the *correct* relationship. On the other hand, Figure 6b convinces students $\overline{DE} \parallel \overline{AC}$, but not that $DE = \frac{1}{2} AC$.

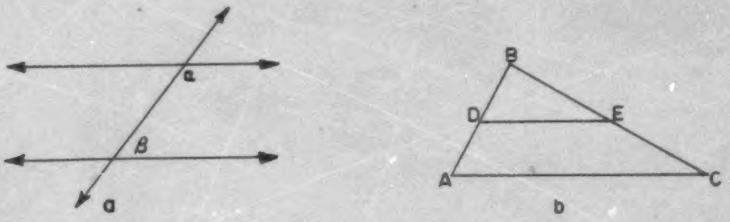

Figure 6

The last two categories seem to support the lack-of intuition theory.

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[7] Here it appears that the proof of the theorem is not convincing. What is revealing about the first example is that the other half of the theorem, viz., that the segment is *parallel* to the third side, *is* used. So the theorem does not seem to entirely agree with "truth" as the student perceives it.

V. Difficulties with orientation. Although these apply only to a very few students, they are still revealing. Three examples are:

A. In Figure 7a, the line is not straight because it is neither horizontal nor vertical.

B. In Figure 7b, the angle is not a right angle because the vertex is on the left.

C. In Figure 7c, the angle is not a right angle because the sides do not have a horizontal-vertical alignment.

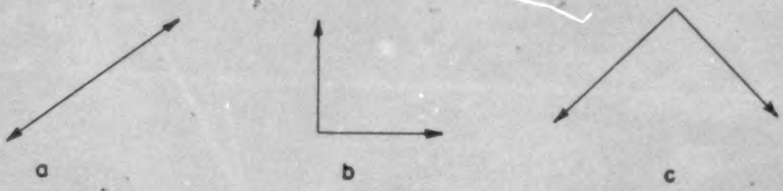

Figure 7

VI. Other common misconceptions:

A. Big triangles have bigger angles than small triangles. Thus if the sides of one triangle are twice the length of the sides of another, the angles have the same ratio.

B. The diagonals of a parallelogram always bisect the angles.

C. The diagonals of any parallelogram are congruent.

D. The bisector of an angle of a triangle always bisects the opposite side.

E. A median to one side of a triangle always bisects the opposite angle.

F. An altitude to one side of a triangle always bisects that side.

G. It is always possible to draw a line which bisects one angle of a triangle and is also the perpendicular bisector of the opposite side.

H. An arc of a circle is equal to its chord.

I. There are many misconceptions about ratios; in general, students have trouble setting up correct ratios between sides of similar triangles.

Difficulty with writing proofs has already been mentioned. What is apparent, however, is that no amount of instruction devoted to the theory of proof construction will guarantee that a student will give a correct proof if, for example, he believes that the diagonals of a parallelogram bisect the angles. The above list of mis-beliefs and dis-beliefs, then, displays factors which can effectively interfere with a student's achievement in high school geometry. A natural question is whether the study of geometry in the elementary school might contribute to the development of this thing called "intuition."

Nearly all of the twenty-five difficulties just listed involve a relation in the mathematical meaning of the word; for example;

is parallel to,
is perpendicular to,
is supplementary to,
is complementary to,
is in the same ratio as,
is congruent to,
bisects.

Apparently some relations are obvious--a scalene triangle, for example, has a longest side, and the segment joining the midpoints of two sides of a triangle is parallel to the third side. But some are not so obvious. Perhaps the implication for the elementary school is that the focus be shifted from properties to relations.

Properties are descriptions of point sets which serve to qualify (or disqualify) those sets for class membership. Thus, we have such properties as,

having x number of sides,
being a polygon,
being a polygon with x number of sides,
having a measure of 90°,
having measure less than 90°.

Relations, on the other hand, always involve pairs of point sets, of which "belonging to the same class as" is a fairly simple example. One of the advantages of the Erlanger Program is that certain relations are invariant when the space is transformed by any member of some group of transformations. Thus, under the topological group, the rectangle labeled X in Figure 8 may look like any of the figures on the right but can never look like a figure on the left. Under the group of similarity transformations the rectangle labeled X in Figure 9 may look like any of the figures on the right but cannot look like any of the figures on the left.

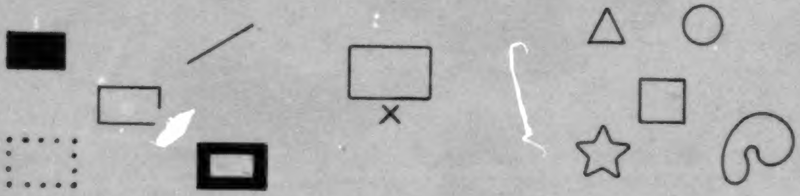

Figure 8

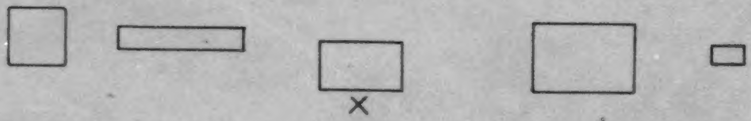

Figure 9

To explore the relation, "Y is the image of X," the student must be attentive to some perceptions and ignore others. In something of the Gestalt sense, the invariant becomes the figure, the irrelevant features the ground. In mathematical terms, under the topological group, for the relation "Y isthe image of X" (the rectangular set X shown in Figures 8 and 9) to hold, it is necessary and sufficient that Y be a simple closed curve. Under the group of similarity transformations, "being a simple closed curve" is necessary, as is "having four sides and four right angles." But neither condition is sufficient--sufficiency requires the proportionality relation for sides. Furthermore, it is not essential that the viewer know the words, "rectangle," "right angle," "simple closed curve," or "proportion," in order to recognize point sets which could be the image of X. As suggested in the previous section of this paper, the transformations are easy to model using hands-on materials, so that the student can observe results. Then with the right sort of questioning he can predict whether some other figure could or could not be an image. The importance of the relation is thus established as an integral part of the problem situation, and need not be introduced in isolation because its importance will be understood "later" or because "that's what mathematicians study."

"Intuition" is one of those undefined terms, like "mathematical maturity," which is probably well understood in the trade, but which is hard to describe. Whenever used, however, it does seem to include such characteristics as ability to imagine circumstances different from those given, and to predict what would happen under the new circumstances; for example, being able to predict what would stay the same and what would change if the figure were larger, or smaller, or of a different shape. Such ability would seem to be what is needed in high school geometry to forestall such unwarranted conclusions as "a bisector of an angle of a triangle always bisects the opposite side."

We tend to think of geometry as a mathematical model of space, and perhaps the appeal that Euclidean geometry has long enjoyed is that it characterizes space as we perceive it. But that means that the relations we recognize as being "true" turn out to be valid consequences of the axioms. However, geometry cannot have that appeal if we fail to recognize certain relationships as being true. Thus a student is unlikely to find high school geometry very appealing if he is lacking in the kind of intuition that tells him, "but of course, that has to be."

Summary

The history of geometry has been unique in mathematics in that it was formalized as a deductive system very early. Thus if there were intuitive roots such as characterize the history of the study of number, these have been lost in antiquity. Furthermore, geometry has not always been viewed as being as necessary as arithmetic in everyday life. Yet geometries constitute an honorable branch of mathematics and deserve a place in the education of children.

The rise of the non-Euclidean geometries showed that a theorem is no more true than the axioms on which it is based. Hence it makes little sense to enquire which geometry is "true"; it does, however, make sense to enquire which geometry (or which approach to geometry) would be more pedagogically sound for children.

Some students study geometry in high school, and for these students some articulation between the content of high school geometry and that of the elementary school would seem desirable. At the present time, the high school geometry course is under attack, with heated debates among the proponents of a vector approach, a transformation approach, an electic approach, and those who favor the traditional course. Yet regardless of how the matter is resolved, Euclidean space will most likely be the central core. Thus anything the student already knows about the nature of Euclidean space will be of help. Facts are nice to know; vocabulary is also nice to know. But what is even nicer is to be able to visualize altered circumstances and arrive at a sound conjecture about what "has to be the case." As a matter of fact, this is nice to know even if you never take another course in geometry.

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Piaget's Thinking
about the
Development of Space Concepts and Geometry[1]

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> For the genetic espitemologist, knowledge results from continuous construction, since in each act of understanding, some degree of invention is involved . . .
>
> Jean Piaget

The conceptual revolution for psychology and education required by Piaget's epistemology is, I think, more appreciated than understood. That is, Piagetian ideas pass around as common currency, among researchers and practitioners alike, but all too often turn out to be only vague facsimiles or counterfeit copies. If we are to benefit fully from Piagetian ideas of psychological development and learning of mathematical concepts, we must translate those ideas with as little transformation as possible. In this paper, I hope to do just that in the context of the purpose of this workshop, i.e., review selected theoretical and methodological issues relevant to research into the development and learning of space and geometry concepts. Specifically, I will (a) review certain critical features of the epistemological and theoretical aspects of Piaget's positions vis-à-vis the development of space concepts, (b) review the available evidence concerning the construction of the "permanent object" which is the fundamental invariant of our spatial world, and (c) summarize Piaget's early work on space and geometry, and (d) offer some methodological suggestions and guidelines for inquiring into cognitive development in general and space and geometry in particular.

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First, as I argued in an earlier paper (Smock, 1973), a coherent, accumulative body of "facts" based on psychological research on the development of space is necessary but not sufficient for building a theory of instruction. Our observation and interpretation of facts are heavily prestructured by our epistemological belief and knowledge base or cognitive structures. Understanding of the relevance and implications of Piagetian concepts and data for education depends on each of us starting at the beginning--i.e., with his theory of the nature of knowledge and knowledge acquisition and then building up our personal knowledge base of empirical findings and demonstration of ideas generated from appropriate experimental settings. Until recently, American researchers' contribution to an accumulative body of knowledge relevant to Piaget's theory was limited to replication of Genevan studies. Current research indicates that the methodological implications (of which I will speak later) of the theory are more clearly recognized (e.g., Forman, 1973; Overton & Jackson, 1973; Toussaint, 1974). But, in any case, the mathematics education researcher needs to build his understanding of the implications of psychological research findings on a firm understanding of the epistemological considerations that are the foundation of Piaget's theory of cognitive development and learning.

Piaget is characterized as a "natural genetic epistemologist." The adjectives may be interpreted with little ambiguity, i.e., "observations of the origins" (of knowledge). However, the specific epistemological position of Piaget is not so readily located, even from his own writings, but there are strong indications that he represents a "radical constructivist" view (Smock, 1973; Smock & von Glasersfeld, 1974; von Glasersfeld, 1975). That is, the environment is, and must remain, a "black box." All we can ever "know" is our own cognitive structures. Knowledge is no more, and no less, than constructed invariants of organism-environment relations, but this construction

> involves the adjustment of, for instance, percepts to conceptual structures which the perceiver has already assembled; and this adjustment of the new to the old is called 'assimilation.' But cognitive equilibration also involves the adjustment of concepts to percepts, and this second type of adjustment, which can take the form of creating a novel structure or of combining several already assembled structures to form a larger conceptual unit, is called 'accommodation.' (von Glasersfeld, 1974, p. 4)

The process of development and learning (cognitive reorganization) is then, in response to conflict among internal functional structures (Smock, 1969, 1974).

The genetic epistemology of Piaget assumes that spatial concepts are constructed through commerce with the perceived environment and are only one aspect of the development of general cognitive structures. Expectedly, then, three themes characterize the Genevan research on space. First, the primary concern is on conceptual and representational space not perceptual space. Thus, acquisition of concepts of the spatial world(s) is a product of general intellectual development. The genesis of space perception is treated separately; the series of experiments on perception is used to highlight the contrasts of perception with that of space representation.

A second theme emphasizes that spatial representations are built up through the process of organization of actions and/or logico-mathematical experience. Initially these are sensory-motor actions (resulting in "practical" space) which later are internalized actions that culminate in operational (infra-logical) systems. The active manipulation of the spatial-temporal environment (objects, empty space, intervals, duration, movements), rather than a passive copying by the perceptual system, generates representations of space.

Finally, a third theme is the characterization of spatial concept acquisitions according to the type of geometric concepts involved--topological, projective, or Euclidean. According to Piaget the historical order (Euclidean, projective, topological) is reversed logically and ontogenetically.

Object Permanence

The construction of space representation emerges at the beginning of the concrete operational period (ages 5-7) from preceding preoperational cognitive acquisitions of the "practical group of displacements" and the even earlier construction of the sensori-motor period. In the beginning, there is an "object"--the origin of, and basic unit for, the development of spatial relations as well as the starting point of Piaget's analyses of the nature of knowledge.[2] Thus, an understanding of the development of object permanence is essential for "knowing" Piaget's epistemology as well as his psychological theory of the concepts of space.[3]

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[2] Citations of Piaget's work will be used only as necessary to identify specific sources of ideas (e.g., quotations). A selective set of primary sources is listed in the references.

[3] See E. von Glasersfeld's (1974) discussion for more details on the epistemological aspects of the notion of the constructed permanent "object."

The infant is at first not aware of any permanent objects, but merely "perceptual pictures," which appear, disappear, and perhaps reappear. In its simplest form, attainment of object permanence means that the infant knows the object continues to exist when it is outside of the perceptual field. The indicator of this knowledge (object permanence) consists in "true search" on the subject's part. A true search is a search for the vanished object independent of subject's on-going actions and perceptions.

The simplest procedure Piaget used to test the "object permanence" is as follows:

1. E shows the child the object (e.g., a doll).
2. E grasps the object in his hand so that the child no longer sees the object.
3. E puts his hand under a coverlet.
4. E withdraws the hand closed.
5. E extends the closed hand to the child.

If the child opens the hand, finds it empty, goes on to search for the object under the coverlet and gets it, then he is considered to have attained the concept of object (i.e., object permanence).

The "logic" supposedly necessary for the attainment of object permanence by the child is as follows:

1. There is an object.
2. A and B are the only possible places the object could be.
3. The object is not a A.
4. Therefore, the object must be at B.

According to Piaget, the infant's conception of external objects as permanent, independent entities is acquired in six distinct stages during the sensory motor period (0-2 years). Evidence for the stages was based primarily on Piaget's observation of his three children's reactions to objects which disappear from view. The behavior pattern characteristics of each stage, along with the estimated age ranges reported by Piaget, are summarized below.

During Stage I and II the infant has the ability for recognition, has intercoordination of schemata, and shows simple expectations. However, the ability to recognize the mother's face, or to look at the object from which the sound comes from, or to continue to look at the place where the object has just vanished and to continue with the sucking response after removal of the nipple, is not a manifestation of the attainment of object permanence.

Recognition

Conceptually, recognition of an object means that the S knows that object he sees now is the same object that he saw previously. Recognition is operationally identified by the fact the subject can respond in the same way when the same object is presented at two time points. For Piaget, recognition is at first only a particular instance of assimilation: The thing recognized stimulates and feeds the sensorimotor schema which was previously constructed for its use. In order that the recognized representation or "picture" become an externalized "object," it must be dissociated form the action itself and the causal relations dependent on the immediate activity.

Intercoordination of Schemata

From the second month of life and the beginning of the third, the child tries to look at the objects he hears, thus revealing the relationships being established between sounds and visual pictures. Does this mean that by presenting certain sounds, the anticipation of a certain image of an object is elicited and thus that the child has already an object concept? Piaget argues that simple intercoordination of schemata between sight and hearing, at the outset, does not generate an objective identity of the visual image and auditory image, but simply a subjective identity, i.e., the child tries to see what he hears because each schema of assimilation seeks to encompass the whole universe. Discovery of the visual picture announced by the sound is only the extension of the act of trying to see. It is not the same case as that of an adult when his act of searching with the glance is accompanied by a belief in the firm existence of the object looked at.

Simple Expectation (Anticipation)

True search is an indication of the beginning of the object concept but, again, simple expectation is not. Simple expectation refers to those behavior patterns in which the search for vanished objects is only a continuation of the earlier act of accommodation. The child only preserves the orientation generated by the earlier perception, e.g., in the case of the disappearing visual image, the child limits himself to looking at the place where the object vanished. If nothing reappears, he quickly shifts attention.

In simple expectation, the vanished object is not yet for the child a permanent object which has been moved, it is a mere image which re-enters the void as soon as it vanishes, and emerges from it for no objective reason. True search is an active search and includes intervention of movements which do not solely extend the interrupted action. In this case, the child will find out where the "thing" could have been put, he will remove obstacles, change the position of the presenting objects at hand, and so on.

One could, perhaps, argue that the child's failure to engage in active search (i.e., simple expectation) is due to lack of motor skill at this early age. However, if the child, while not knowing how to search (motorically) for the absent object, nevertheless believes in its permanence, then "true search" should begin as soon as prehension skills have been acquired. Such is not the case as the child's behavior in Stage III indicates.

Stage III is a transition period from prehension of an object at hand to true search for a missing object. In this intermediate stage, five types of behavior are distinguishable: (1) visual accommodation to rapid movements, (2) interrupted prehension, (3) deferred circular reaction, (4) the reconstruction of an invisible whole from a visible fraction, and (5) the removal of the obstacle preventing perception. The first of these behavior patterns merely extends those to the second stage, and the fifth fulfills those of the fourth stage.

All the behavior patterns enumerated hitherto merely extend the action in progress. Clearly, visual accommodations to rapid movements, interrupted prehension, and deferred circular reactions, all consist merely in returning to the momentarily suspended act--not in complicating that action by removing the obstacles which arise. The reconstruction of invisible totalities and the removal of obstacles preventing perception both seem to involve such differentiation, but this only appears to be true. That is, when the child tries to get to a half-hidden object and, to do this, removes the obstacles which cover the hidden portion, the action is by no means as complicated as that of removing a screen masking the entire object. In the latter case, the child must momentarily give up the attempt at direct prehension of the object in order to raise a screen which is recognized as such. On the contrary, in the former case, the child sees part of the object and tries to grasp it; he only reconstructs the totality as a function of this ongoing action and not because a new action pattern, consisting of removing the screen, has been differentiated. Removal of obstacles preventing perception requires knowledge of an object in relation to the subject and not in relation to the object, i.e., the obstacle-screen and the object as such are not yet related. From this point of view, the object is still only the extension of the action in progress.

True Search

Stage IV marks an important transition. Prior to Stage IV, the infant lacks object permanence and knows a thing and its location only in the context of his ongoing actions. He either cannot find hidden things or can only find them when he has begun to reach for them before they disappear. In Stage IV, the infant is aware of the object's permanence. If he observes an object disappear, he searches for it even when he has not begun to reach for the object before it disappeared. However, his objectivity has an important limitation which indicates that his ideas of things continue to be bound up with how he acts upon them. An object is not localized in terms of where it has moved, but in terms of where it was found in the past.

Piaget employed the following procedure to test children for Stage IV behavior. First an object was hidden in place A and then in place B. Piaget noted that after the children had found the object at A, they attentively watched the object being hidden at B. However, when the object disappeared in view, the children searched at A. This pattern of success at A and failure at B was referred to as $A\bar{B}$.

Three possible explanations are offered for the $A\bar{B}$ behavior: difficulties in memory, in spatial localization, and in object conceptualization. Piaget (1954) makes it clear however, that the first two are considered but different aspects of the difficulty in object conceptualization: "Faced with the disappearance of the object, the child immediately ceases to reflect: in other words, he does not try to remember the sequence of positions and thus merely returns to the place where he was successful in finding the object the first time" (p. 61). The memory explanation is simply a way of saying the child has trouble keeping track of places, and the second, or spatial localization explanation, provides reasons for this difficulty. Further, Piaget argues that the infant localizes objects in terms of a scheme based upon the infant's prior actions. Accordingly the $A\bar{B}$ error occurs because there has been no previous action at B, and therefore the infant has no "memory" of that place. On the other hand, he is able to localize A because it was at that locale that his practical action brought him a toy. When he sees a toy being hidden at B, he registers only that a toy is being hidden at a "place" and so searches at the only place he is able to localize.

When talking about the third explanation, Piaget points out that adults are able to think of particular objects only because of the assumption that objects are independent of the many places they may occupy. However, if we did not distinguish thing from place, the adult would be aware of "ball-under-the-arm-chair," "ball-under-the-cushion," etc., which is what the infant does in Stage IV. The child endows objects with only a few special positions without being able, consequently, to consider it as entirely independent of them. Piaget (1954) concludes: "In a general way in all the observations in which the child searches in A for what he has seen disappear in B, the explanation should be sought in the fact that the object is not yet sufficiently individualized to be dissociated from the global behavior related to position" (p. 63).

The first acquisition of Stage V is signified by success on a task where an object is hidden under a first screen which the child finds then under a second screen, and the subject no longer searches for the object under the first screen, but only under the second one. As mentioned above, the child succeeds in keeping track of only the visible displacements of the object and locates it only when he has actually seen it. From repeated experiments, Piaget found that when an invisible displacement of object intervenes, the child relapses into the same difficulties which he has already overcome when visible displacements were involved. These findings furnish us with a good example of the law of "temporal displacement," i.e., when an operation passes over from one plane to another, it has to be relearned on this new plane. In particular, the groups of displacements of object which, at the beginning of the fifth stage, had been constituted on the plane of direct perception of relationships of position, must be formed anew as soon as it has been transferred to the plane of representation of these relationships.

In the final (VI) stage, the object is not as it was during the first four stages, i.e., merely an extension of various accomodations. Nor is it, as in Stage V, merely a permanent body in motion whose movements have become independent of self but only to the extent and scope of perception. Instead, in Stage VI the object is definitely now freed from perception and practical action.

It should be emphasized that a subject's search for an object under a screen, after the subject has seen the object disappear under the screen (Stage IV and V), does not necessarily presuppose that the subject "imagines" the object under the screen. Rather, the search simply indicates that the subject understood the relation of the two objects at the moment he perceived it (at the moment the object was covered) and, therefore, interprets the screen as a sign of the actual presence of the object. It is one thing to assume the permanence of an object when one has just seen it or when some other object in sight recalls its presence. It is quite another thing to imagine the first object when there are no perceptual signs to confirm its hidden existence. True representation begins only when no perceived sign commands belief in permanency, that is, from the moment when the vanished object is displaced according to an itinerary which the subject may deduce but not perceive. With regard to the infant at the fifth stage, the objects are not "permanent" to the extent that he does not know how to imagine or to deduce the invisible displacements of bodies as objects truly independent of the self. On the contrary, the representation and deduction characteristics of the sixth stage result in disassociating the object from action and perception and objects in motion become real objects independent of the self.

Review of Recent Studies

A review of studies of object permanence revealed that most are concentrated on Stage IV, because Stage IV is considered by Piaget to be the critical period where the "true search" behavior begins. The procedures used in these studies generally are the same as those Piaget used except for additional control of critical variables such as the length of interval between hiding the object and permitting the child to search (usually a 3-second delay was used), and the number of times the object is hidden at A before it is hidden at B (usually the object was hidden twice at A before it is hidden at B).

Gratch and Landers (1971) observed infants biweekly until they were between 6 and 12 months of age in order to replicate and elaborate on the observations Piaget employed to define State IV of object concept. Their observations suggest that the Stage IV phenomenon is part of an age-related sequence of responses to hidden objects. Before the infants found an object hidden out of their reach, they were able to find it when partially hidden (P) and were able to find objects that were grasped before they were covered (G). The Stage IV error ($A\bar{B}$) phenomenon occurred repeatedly over a period of 1-3 months. During that period, however, the character of $A\bar{B}$ changed. Initially, infants seemed to ignore the displacement of the object to a second position. Later, they appeared to be in conflict over whether to use the cues provided by their prior successful searches or their awareness of the displacement of the object. Finally, they came to rely on the cue of object displacement.

Landers (1971) studied the effects of "active experience" (i.e., searching for the hidden toy) versus "passive experience" (i.e., observing but not searching) on the $A\bar{B}$ behavior.[4] Using infants between the ages of 7½ and 10½ months, he found that infants who had more experience finding an object at the A position tended to have more difficulty finding the object when it was hidden at the B position. This suggests that simply watching the experimenter hide and uncover a toy at A does not establish the A side as a "special" place, which active search does. The results seem to clarify and establish empirically Piaget's argument that the Stage IV infant's behavior and object concepts are dependent upon "context" and "action." The previously reinforced motor response seems to be a more potent aid to memory than the most recent visual input when representational processes are just beginning.

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[4] Three groups were used: (GI)--"low active A-experience group"--two active experiences at A; (GII)--"high active A-experience group"--10 active searchings at A; (GIII)--"high passive A-experience group"--two searching trials at A plus 8 observing trials.

Evans and Gratch (1972) studied the $A\bar{B}$ error to evaluate the relative merits of place (spatial localization) and thing-of-place (object conceptualization) arguments Piaget offered for explaining the $A\bar{B}$ error. Infants between the ages of 8½ and 10½ months served as subjects. Half of the infants were assigned to "same toy" condition and the other half to "toy change" condition in which different toys were used on B-place test trials than those used on the A-place trials. The result showed no difference between subjects' performance on the two conditions. This result, Evans and Gratch concluded, supported the notion that the $A\bar{B}$ error occurs because A has somehow become "a place where hidden toys are found," rather than because, as Piaget has argued, the particular object belongs at A. If Piaget's thing-of-place argument were correct, then the fact that the child sees a new object hidden at B should increase the probability that he will search correctly since no previous action by the child has endowed the new object with a place.

Lecompte and Gratch (1972) investigated directly the development of object identity in infants by varying the objects instead of its spatial position. This involves tricking the child by hiding one thing and having him find another. The assumption is that if a child is aware of the permanent nature of the object being hidden, in the sense in which adults are and which Piaget attributes to infants at Stage IV, he will be surprised when he is tricked because his belief in permanence will have been violated. Subjects consisted of infants at three different age levels: 9, 12, and 18 months. Two rating scales, both with seven categories were devised to evaluate the child's immediate reaction to the trick (levels of puzzlement) and his subsequent behavior (what he was surprised or puzzled about). The result showed that infants at all age levels reacted differently on the trick trials than they did either before of after the trick (two trick trials were inserted among a total of nine trials). Older infants reacted with high puzzlement and searched for the missing toy and for the cause of the disappearance. Younger infants were mildly puzzled and only focused on the new toy. The authors concluded that these results conformed to Piaget's account of the development of the object concept.

As has been mentioned previously, Piaget has not controlled the time interval between object being hidden at A and the time the subject is allowed to search. Harris (1973) studied the role of delay and compared the subjects' performance on a 0-sec. and 5-sec. delay task. Infants 10 months of age served as subjects. The results indicated that infants at this age search correctly, but errors are more likely with delay and if cues previously associated with finding the object distract the infant after its disappearance. Harris draws attention to the similarity between this behavior of the human infant and the frontally lesioned primate. He proposes that maturation of frontal cortex may be important for the development of search behavior. The argument is based on the fact that Harlow, H., Harlow, M., Rueping, and Mason (1960) found that prior to 5 months, training on delayed-response tasks led to little or no improvement. After 5 months, learning was increasingly more rapidly with increasing age. The learning curves for 0- and 5-sec. tasks were similar, but the ability to solve the delay task developed more slowly. Harris (1973) concludes that these data demonstrate that the infant's immaturity makes him susceptible to proactive interference in short-term memory.

Webb, Massar, and Nadolny (1972) observed the behavior of 14- and 16-month-old children in searching for hidden objects. A three-choice, instead of two-choice, task was used, and the subjects were allowed to search for the object on each trial until it was found. The major finding was that the second-choice behavior of the 14-month-old children was essentially random but that of the 16-month-old children was predominately correct after an initial error. The interpretation of these authors was that the child knows the location of the object in two distinct and relatively independent ways. That is, he remembers both past actions in locating the object, and the location of where he saw the object hidden. In the search, first credence is given to the past location of the object whereas the immediately prior perception becomes functional only after the other cues prove unreliable. Thus, the child knows the correct location of the hidden object in some sense even while making the error on the initial choice. The interpretation applied to these data has an interesting analogy to several recent attempts to reformulate Piaget's conservation problems, i.e., that perceptual strategies interfere with the child's use of other solution processes. Webb et al. suggest that an overdependence on action-marked cues and past success overrides a functional memory for visually presented events.

Gruber, Girgus, and Bannasisi (1971) modified Piaget's methods of studying object permanence in children in order to study development of object concepts in cats. Eight behavioral tasks were constructed: (1) an auditory stimulus (click) off to one side; (2) an object swung in a circle around the kitten; (3) an object placed in front of the kitten and then moved slightly; (4) an object and kitten placed on a stool; (5) a kitten playing with an object and auditory distraction introduced; (6) a kitten playing with an object and visual distraction introduced; (7) a kitten playing with an object and object is covered while kitten is distracted; (8) an object is covered while the kitten is playing with it. The data indicated that cats reach only an early "developmental" limit. Unlike children, the kittens were unable to follow an object through a series of invisible displacements. Interestingly, house-reared cats showed similar limitations but advanced more rapidly than cage-reared animals. Finally, the longitudinal data suggested that cats go through four stages, rather than the six found in children. In 24 weeks kittens develop as far as children do in the first year, but the child's behavior eventually becomes more complex and more general.

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Uzgiris and Hunt (1966) constructed a series of scales for assessing infant psychological development. For each of the series of "eliciting situations," certain infant actions were selected as indicative of significant steps in that branch of psychological development.[5] To give a concrete idea of the scales, the "noticing the disappearance of a slowly moving object" item is a good example:

Noticing the disappearance of a slowly moving object.

Location:	The infant may be supine on a flat surface, in an infant seat, or sitting up by himself.
Object:	Any bright object that attracts the infant's attention.
Directions:	Once the infant has focused on the object, move it slowly to one side and away from the infant, making it disappear below the edge of the infant's seat or the surface on which he is placed. After a few moments, bring the object back in front and slightly above the infant's eyes from the opposite side. Always move the object in the same direction and have it disappear at the same point.
Repeat:	3-4 times.
Infant Actions:	a. Does not follow object to point of disappearance. b. Loses interest as soon as object disappears (eyes begin to wander and then focus on any interesting object within view). *c. Lingers with glance at the point where the object has disappeared. *d. After several presentations, returns glance to the starting point or the point of reappearance before the object has reappeared. e. Searches with eyes around the point where the object has disappeared.

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[5] Escalona and Corman (1967) also constructed a scale for assessing the degree of object permanence. The administration condition and responses are described which enabled the examinar to score the child as belonging to a particular level of a particular stage.

*Actions marked with an asterisk (*) are considered critical for this particular step of development.

Miller, Cohen, and Hill (1970) replicated the ordinal scales of object permanence constructed by Uzgiris and Hunt with 84 infants (ages 6, 8, 10, 12, 14, 16, and 18 months). Consistent age changes were found that suggested two overlapping developmental dimensions: (1) the ability to deal with visible versus invisible displacements and, (2) the ability to handle nonsequential versus sequential displacements. However, the trend of data suggested, contrary to previous evidence, that the infant can cope with single invisible displacements not involving movement before he can handle complex visible displacements that do involve movement.

Bell (1970) studied the development of the concept of object as related to infant-mother attachment with infants between the ages of 8½ and 11 months. Two scales, closely comparable to those constructed by Escalona and Corman (1967) and Uzgiris and Hunt (1966) were devised. Seventy percent of the infants showed "positive décalage," i.e., they tended to be more advanced in the development of "person" permanence than in the development of the concept of inanimate objects as permanent. Another important finding was that the development of the object concept was intimately associated with the attachment of a baby to his mother. The babies who manifest strong evidence of "décalage" displayed more active efforts to establish and maintain interaction with the mother through approaching, reaching and/or initiating interaction.

Bower (1971) has raised the question as to whether object permanence could be a built-in property of the nervous system (as is the case with so many other kinds of perceptual knowledge). During the experiment with 20, 40 and 80 day old infants, a screen moved in from one side and covered the object. After various intervals (1.5, 3, 7.5, or 15 sec.) the screen was moved away. In half of the trials the object remained as the screen moved away, and in the other trials the object was removed prior to removing the screen. The results showed that when the object had been occluded for 1.5 sec. all the infants manifested greater surprise at its nonreappearance than at its reappearance. The index of surprise here was determined by a change in heart rate. Thus, it would seem that the infants expected the object to be present. However, while the oldest infants "expected" the object to reappear, i.e., showed the greatest heart rate deceleration after the longest occlusion period, the youngest infants exhibited a reverse effect after the longest occlusion period, i.e., showed more surprise at the object reappearance than at its nonreappearance. It seems that even very young infants know that an object is still there (built-in structure) after it has been hidden, but if the time of occlusion is prolonged, they forget the object altogether.

Bower (1971) suggested that while the older infants identify an object by its features, the younger infants (less than 16 weeks) identify objects by place or movement. To test this idea, a group of infants between 6-16 weeks and a group between 16-22 weeks were examined in four situations: (1) An object moved along a track, went behind a screen, emerged on the other side, moved on for a short distance, stopped and then returned to its original position.

(2) The object moved along the track, went behind the screen and at the moment when the object should have emerged on the other side of the screen a totally different object emerged, moved on for a short distance before reversing and repeating the entire cycle in the opposite direction. (3) The object moved along a track as before, except at the time when according to its speed before occlusion it should still have been behind the screen, an identical object moved out. (4) The object moved along the track as before, and at the time when it should have been behind the screen a totally different object moved out. The results are summarized in Table 1.

Table 1

Infants Reactions to Bower's
Object-Movement Situations

Conditions		Younger Infants (6-16 weeks)	Older Infants ([illegible]-22 weeks)
(1)	One object, one movement	Continued to follow the path of movement when the object stopped	Stopped tracking the object when it stopped
(2)	Two objects, one movement	(Same response as above)	25% of the times infants would look to the other side of the screen when the object stopped
(3)	One object, two movements	Upset and refused to look any more; infants did not continue to follow the object's path when it stopped	On every trial, infants would look to the other side of the screen when the object stopped
(4)	Two objects, two movements	(Same response as above)	(Same response as above)

From these results, Bower argues that the younger infants responded not to moving objects but to movement per se. Similarly, their responses were not to stationary objects, but to places. On the contrary, older infants identify an object by its features rather than by its place or movements, for them different features imply different objects that can move independently.

Conclusions

Object permanence means that an infant knows the object continues to exist when it is outside of the perceptual and action field. "True search" is the indication of object permanence. By true search Piaget means that the subject engages in an active search for the vanished object independent of his action and perception.

Six stages in the development of the concept of object permanence have been identified with Stage IV marking an important transition. Prior to Stage IV, the infant knows a thing only in the context of his ongoing actions; he is concerned with his objective as one and the same thing as his desire. In short, before Stage IV the action is the source of external world. After Stage IV the object becomes detached from the infant's activity and gradually acquires an independent status.

Recent studies of object permanence have concentrated on the infant's Stage IV concept of object and, in particular, on the $A\bar{B}$ phenomenon. $A\bar{B}$ means that after the infant has experience of successful searches at place A, though he has watched tne object being displaced to place B, his search for the object is at A. The implication is that the infant conceives the object as being the product of his action: The toy does not exist by itself, but is the end result of the reaching of his hand at A. At the end of Stage IV, the infant's search for object begins to oscillate between place A and B, and this is the mark of the beginning of the object being detached from the action. Studies of infant concept of Stage IV generally have confirmed the developmental sequence observed and theoretical interpretations reported by Piaget.

Landers' (1971) study showed the influence of prior motor experience on the subsequent search behavior of the infant. Evans and Gratch (1972) argue against Piaget's thing-of-place interpretation of $A\bar{B}$ behavior but do not take the role of action into consideration. The "thing-of-place-A" concept of object that Piaget offered is simply the result of the "action-at-place-A." If the subject has only had the experience of "action-at-A" there is no reason to expect that the infant would search at B even when the infant has noticed the object being hidden at B is different from what was hidden at A in the former trials.

The critical point in Lecompte and Gratch's (1972) study is that while the older infants were engaging in a search for the missing toy, the younger infants were only focusing on the new toy. The younger infants' behavio only indicated that the "toy change" had been noticed, or, the ability to identify the object by its features had already been demonstrated. Bower's (1971) study showed that the ability to identify the object by its features was present at the age of 16-22 weeks. Since this behavior does not fulfill the criterion of "true search," there is no reason to believe the object still exists for the infant once it vanishes from the infant's perceptual field.

Harris (1973) suggests a physiological basis for lack of object permanence. For him the inability to solve delayed reaction tasks is the basic source of difficulty for infants. It might be possible, as he indicates, that the ability to solve such delay tasks is related to maturation of frontal cortex.

Webb et al. (1972) used a new procedure, in which the infants were allowed to correct errors and make a second choice on a three-choice task. The results of the study showed predominant correct second choice of the 16-month-old infants. It is clear that 16-month-old infants do not forget prior relevant perceptual input; the error on the first attempt only shows that, at this age, "action" overrides prior perceptions at this cognitive developmental stage. It seems, therefore, that the development of any conservation (object, area, weight, etc.) goes through the same process with action being predominant initially, then perceptual factors and finally the representational (logico-mathematical) system.

Gruber et al. (1971) demonstrated that cats never go beyond the Stage IV concept of object. This result seems to suggest that there is a corresponding change in nervous system (e.g., in frontal cortex) along with the development of object concept.

The scales that Uzgiris and Hunt (1966) and Escalona and Corman (1967) developed to standardize tests of infant's object concept should prove valuable. Piaget always allows great latitude of procedures in experiments and in many cases it is the best way to find out about what a child knows. However, standardization of procedures are, at least, necessary for comparative analyses.

Bell (1970) found that the development of person permanence was more advanced than that of inanimate object concepts. Though Bell did not emphasize the role of experience in the development of object permanence, infants who had more interaction with the mother were significantly more advanced in the development of person permanence than those with less interaction. Piaget (1954), in interpreting the transition from Stage IV to Stage V, emphasizes the role of such experience.

Construction of Space

The analysis of the nature of space has preoccupied philosophers and scientists for centuries. Piaget and Inhelder (1967) were intrigued by the fact that the historical and logical sequences of geometry (measurement of space) were in conflict. Geometry primers are almost unanimous in presenting the fundamental ideas of space as resting upon Euclidean concepts such as straight lines, angles, squares, circles, measurements, and the like. On the other hand, abstract geometrical analysis tends to show that the fundamental spatial concepts are not Euclidean at all, but 'topological', i.e., based entirely on qualitative correspondences involving concepts like proximity and separation, order and enclosure.

With his studies of the child's conception of space, Piaget has successfully demonstrated that the child's space invariably begins with simple (basic) topological types of relationships long before it becomes projective or Euclidean in nature.

The misconception that space concepts begin with simple Euclidean characteristics arises primarily from ignoring the fact that the evolution of spatial relations proceeds at two different levels. There is an interval of several years between perceptual and conceptual construction of space (despite their pursuing a similar path of development). Children's perceptual space has reached projective and quasi-metric levels during the first year of life, when their conceptual space has barely begun.

The present review concerns only conceptual space. But since the perceptual and sensori-motor structures constitute both the point of departure and the foundation of the entire conceptual construction of space, we will start by going over briefly the development of perceptual space as seen in the child's perception of shape.

In the experiments of the child's perception of shapes, Piaget and Inhelder (1967) identified three stages. During Stage I, the only shapes recognized, and drawn, are closed, rounded shapes and those based on simple topological relations such as openness or closure, proximity and separation, surrounding, etc. These relations express the simplest possible coordinations of actions, e.g., following a contour step by step, surrounding, separating, and so on. With Stage II the recognition of Euclidean shapes begins based on the distinction between straight and curved lines, angles of different sizes, parallels, and relations between equal or unequal sides of figures. At this level, the coordinations of actions is of a more complex type in that the child now recognizes a straight line by the action of following--with hand or eye--without changing direction, and recognizes an angle by two such intersecting movements. Finally, at Stage III the child is able to return systematically to a fixed point of reference while exploring an object. That is, he now can coordinate all his movements into a single whole according to a system of reference. For example, if the object is a figure of a starfish engraved on a surface, and the child is asked to identify the figure through haptic exploration, a child at Stage III is able to touch each arm in turn, exploring the angles between the arms, and returning systematically to the center where the arms meet.

Perceptual organization of space proceeds through a developmental stage sequence. At first it is based on topological relationships, and later on projective and Euclidean relations. Does conceptual space, after some year's interval, pass through the same phases?

The most important difference between topological relations and the projective and Euclidean relations is in the way in which different figures or objects are related to one another. Psychologically, topological relations are the most primitive ones; the relations of proximity, separation, order, enclosure and continuity are built up empirically between the various parts of figures or patterns which they organize. These relations are independent of any contraction or expansion of the figures and, therefore, do not conserve features such as distances, straight lines, or angles during changes of shape. Hence, we may say that topological space is purely internal to a particular figure whose intrinsic properties it expresses, and it is impossible for relationships of this type to lead to comprehensive systems linking different figures together. It is in this sense that topological relationships are considered primitive.

Projective space begins psychologically at the point when the object or pattern is no longer viewed in isolation, but begins to be considered in relation to a 'point of view'; the viewpoint of the subject (in which case a perspective relationship is involved) or that of other objects on which it is projected. Because of this property concerning viewpoints, the study of projective space can be called the 'geometry of viewpoints'.

Euclidean space is different from projective space in that the concepts of distance and measurement are introduced. It deals with the orientation of objects relative to each other and to a system of reference points arranged along different dimensions. Because of its concern with the objects as such, the study of Euclidean space may be called the 'geometry of objects'.

In short, topological space deals with the internal relations of the isolated object, projective space deals with relations of objects to the subject, and Euclidean space deals with relations of objects to objects.

The evolution of the conceptual spaces can be clearly demonstrated in children's drawings of geometrical figures. Three distinct stages can be identified from the drawing of children when they are asked to copy figures. During Stage I (0-4 years), the circle is drawn as an irregular closed curve, squares and triangles are not distinguished from circles, and the drawing is of two more or less intersecting lines. While there is no distinction as yet between straight-sided and curved figures, depending on the complexity of the figures, there is correct copying of the topological properties.

Stage II (4-6 years) marks the beginnings of the differentiation of what Piaget designates as Euclidean shapes. Thus, a square is distinct from a triangle, a circle from an ellipse. Two types of crosses are distinguished which marks the discovery of oblique lines. Finally, the rhombus (as distinct from the square)is reproduced correctly.

At Stage III (6 to 7 years), the child's idea of shapes is at the operational level. Children draw figures quickly and correctly and these constructions reflect anticipation through mental images (advanced organization) in terms of potential measurements, coordinations, etc.

Development of Topological Space

The study of drawings has shown that the simplest topological relationships such as proximity and separation also are the first to emerge in the course of psychological development. This order of appearance is also maintained when space is treated axiomatically by geometricians. In the case of a linear series, the relationship of proximity subsisting between separate elements A, B, C is sufficient to provide a basis for the relation of order. This may be perceived intuitively at an equally early stage of development. The notion of order or sequence is thus a third basic topological relationship.

The relation of order exemplified by three elements arranged in a series ABC also entails a specific relationship expressed by the word 'between.' Thus B is between A and C, and at the same time between C and A. This relationship, whose invariance remains a mystery to children who have not yet learned to reverse a series, evolves concurrently with the notion of order itself. The relation between is one particular instance of the more general relationships of 'surrounding'. These are, of course, elementary spatial relationships, just as much as proximity, separation, or order. Indeed, as regards the construction of space, they are even more important, since it is most probably these relationships which lead the child by the most direct route to differentiate and build up the three initial topological dimensions.

If the location of a point between two others designates a one-dimensional surrounding (i.e., a line), and the location of a point inside or outside a closed figure designates a two-dimensional surrounding (i.e., a surface), then the relationship of a point, whether inside or outside a closed box, designates a three-dimensional surrounding (i.e., 3-dimensional space).

It appears that in the case of 'surrounding', there is one area in which perceptual relationships have not yet been developed and hence is ideally suited for studying the main features of representation. Knots have the added advantage of having been the subject of extremely detailed geometrical analysis. From the standpoint of mental development, the knot is something which the child learns to form at an early age, and it is therefore eminently suited to psychogenetic investigation. The tasks consist of asking the child to (1) reproduce an 'overhand' knot (an ordinary single-looped knot), a circle, a figure of eight, or a pseudo-knot (homeomorphic with the circle when the ends of the string are joined); (2) to compare left vs. right overhand knots, taut vs. slack knots, and an overhand knot vs. a circle, a figure of eight, or a pseudo-knot; and (3) to predict the shape of the knot following certain transformations

Stage I (up to 4) can be divided into substages. During Substage IA the children could not copy knots. Either one end of the string is wound around the other without inserting either end in the loop or one end is inserted in a half-loop without superimposing it. In neither case is there the necessary 'surrounding' and, consequently, no knot. During Substage IB the children learn how to copy the knots but are unable to follow the various sections of a slack knot with one finger, nor were they able to distinguish true from pseudo-knots.

At Substage IIA the child perceives the identity between a pair of taut knots, or between a pair of loose knots, but this identity is lost if one of the knots is tightened (or slackened) even though each of the pair is homeomorphic to the other.

A parallelism between the concept of number and the concept of space can be found at this transitional period. At about 5 or 6 years of age, a child can establish intuitively the correspondence between a number of objects and a separate but equal number of other objects, but only when the arrangement of objects produces a similar visual pattern (such as two straight rows). As soon as the intervals between consecutive items in one of the rows is altered the equivalence of number is no longer recognized. Similarly, in the experiment reported above, the child at 5 or 6 can recognize either tight or slack knots when compared with visually identical models, but is unwilling to grant their equivalence without the perceptual equivalence. At Stage IIB the correspondence between the taut and slack knot is established through motor anticipation. For example, the child might say: "If I pull, I'll get the one before." In other words, the child at this stage can imagine the knot in terms of actual transformational motor activites, rather than perceptually static patterns.

Finally, at Stage III (8-10 years), the actions become internalized and completely reversible in nature. Whatever happens to the perceptual patterns, the relationship of the surrounding remains unchanged.

Development of Projective Space

As has been pointed out earlier, topological space only furnishes the basis for that type of analysis which operates from the standpoint of each figural object considered in isolation. In projective space, however, the object is considered in relation to the viewpoint of the subject (perspective) or that of other objects (projective). Thus, projective relationships presume the intercoordination of objects separated in space.

Piaget's reports of his investigations concerned with projective lines and perspective attempt to show that the precondition for forming projective straight lines is a progressive discrimination and coordination of different viewpoints. Next he takes up projection of shadows and provides a demonstration that projection of shadows is understood at the same level of development as perspective transformations relating to the same object. In the study on "viewpoints" (of a group of mountains) the problem of the overall coordination of perspectives, such as arises when the observer moves around and about a number of interrelated objects, is analyzed. The mutual implications of projective and Euclidean space notions are then studied with the rotation and development of surfaces and the distinction between perceptual and conceptual space is considered. This review will be specifically concerned only with the problems of construction of a straight line and coordination of perspectives. The former involves perspective of a single object, while the latter with perspective vis-a-vis a group of objects.

The concept of straight line results from the child's first attempts to relate objects spatially in a system of projective viewpoints or coordinates. Strictly speaking, the topological idea of a line does not include the straight line at all. To transform an ordinary line (the only kind of line recognized by topology) into a straight line requires the introduction, either of a system of viewpoints such as the elements of a line masking each other to form a perspective, or else a system of displacements, distances and measurements.

The task of constructing a straight line will illustrate clearly the critical difference between perceptual and conceptual space. The task consists of asking the child to use match-sticks to form a straight line parallel to the edge of a square table, lying at some angle to the two adjacent sides of the square table, or across a section of a round table.

The results show that at Stage I children can recognize a straight line and distinguish it from a curve, but they are unable to construct such a line parallel to the edge of a table, except when allowed to do so where an existing model or edge of the table is spatially very close. Otherwise, the performance reflects the formation of a topological line with successive elements very close together and curved in various ways. When the line to be imagined and constructed conflicts with perceived straight or curved lines lying adjacent, such as on the ground offered by the table top, the child is no longer able to form a straight line. In this case, the line no longer consists of merely imitating a past or present perception, but entails creating new relationships within an existing pattern distinct from those sought after. Such an achievement requires a projective operation based on the action of 'taking aim', or else a Euclidean operation based on change of position.

At Substage IIA (4-6 years), the child arranges the matches parallel with the edge of the table, and can even arrange them against a neutral ground. But, he is unable to resist the influence of an edge (of the table) if the required line is no longer parallel to it. Two distinct types of spatial concepts need to be recognized at this point. The first, intuitive, is no more than an internalized imitation (a mental image) of previously perceived events. Consequently, it can be either favored or discouraged by current perceptual configurations. The second, not evolved as yet, is based on operations, and therefore is freed from the influence of such configurations. At Substage IIB (6-7 years) the child is able to free himself from the influence of the surrounding perceptual configuration and to form lines independent of the edges of the table. At Stage III (8-11 years) the child discovers a new technique of visual alignment, i.e., the operations of 'sighting' or 'taking aim'. The child discovers the projective straight line, then, when the fact that two points X and Y can be related to the observer O through the agency of the line OXY is grasped. The conceptual straight line thus differs from the perceptual straight line (and from the topological line) by virtue of the awareness of the part played by different points of view. To join X and Y together in a direct line, the child must, at the same time separate them from the perceptual ground, join X and Y either by means of a movement or else by visual inspection. The second procedure can only be carried out be discriminating between different points of view, and it is choice of the point of view OXY that enables the child to correct his alignment.

The perspective of a group of objects as viewed by an observer from different positions, or alternatively by a number of observers, is examined with two aims in mind. First, to study the construction of a global system linking together a number of different perspectives, and second, to examine the relationships which the child establishes between his own viewpoint and those of other observers.

The task used by Piaget consists of three clearly distinguishable model mountains on a pasteboard. The child is asked to imagine, or to identify from pictures, the various views (or 'snapshots') of the group of mountains when it is seen from different positions.

The results show that throughout Stage II (5-7 years) the child does not distinguish between his viewpoint and that of other observers. At Substage IIA, performance is confined to reproduction of his own point of view.[6] In Substage IIB the child makes some attempt to distinguish between different viewpoints, but usually lapses into the egocentric constructions of Substage IIA. The child does not yet think in terms of 'groupings' of projective relations and correspondences or the invariance of the correspondences amid the endless transformations of the projective relationships.

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[6] It is the egocentric illusion which prevents these children from reversing left-right/before-behind relations and thereby rotating the perspectives along with the changing viewpoints, a continuation of the illusion which is responsible for absence of shape constancy in young babies' perception.

At Stage III (7-8 to 11-12 years) the child shows a progressive discrimination and coordination of perspectives. At Substage IIIA certain relationships are varied with changes in the position of the observer, but there is still no comprehensive coordination of viewpoints. In most cases, the before-behind relationship can be reversed, but the left-right relationship retains its rigidity. At Substage IIIB the child achieves a complete relativity of perspectives as demonstrated by the discovery that: (1) to each position of the observer there corresponds a particular set of left-right, before-behind relations between the objects constituting the group of mountains, i.e., the point to point correspondence between position and perspective; and (2) between each perspective viewpoint valid for a given position of the observer and each of the others, there also is a correspondence expressed by specific changes of left-right, before-behind relations. It is this correspondence between all possible points of view which constitutes coordination of perspectives.

From the point of view of both mathematical construction and psychological development, projective and Euclidean space are closely related and both derive from topological space. In addition, projective and Euclidean space are related in another way, i.e., it is possible to construct a series of transitional stages between projective and Euclidean space by considering affinities and similarities. Affinities may be defined mathematically as projective correspondences conserving parallelisms, similarities as affinities conserving angles, and Euclidean displacements as similarities conserving distances.

The conservation of parallels can be demonstrated in the study of reactions in a very simple case of 'affinitive' transformations, namely, the increase and decrease in the width of the rhombuses in a set of "Lazy Tongs" (see Figure 1).

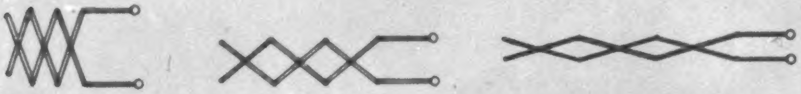

Figure 1. The transformations of rhombuses seen in "Lazy Tongs."

The task consists of asking the child to predict and draw what will happen when the handles are opened or closed.

The results show that the child at Substage IIA (4-5½ years) is unable to anticipate any transformation when the apparatus is stationery. If he sees the beginning of a change, he can imagine the continuation of it, but only in the form of endless enlargement of the 'windows.' At

Substage IIB (5½ - 7 years), the child foresees that the rhombuses will grow longer, and he is also prepared to admit that they will eventually get smaller again. But this prediction remains global and inadequate, for the transformations appear to him not as a continuous (quantitative) series but as abrupt, disjointed changes (qualitative).

At Stage III (7-11 years), all the transformations are predicted correctly, and the necessary parallelism of opposite sides of the rhombus is recognized, i.e., the parallelism of the sides is operationally aligned with the transformation of the figure as a whole and is no longer simply perceived or imagined intuitively.

The next task is to examine the discovery of the similarity of angles, such as in triangles or rectangles, in order to analyze similarities. To do this, children were asked to draw, or compare, pairs of similar or dissimilar triangles.

The results show that at Substage IIA (4-6 years) the child faced with the task of drawing a triangle larger than the model is content with producing any sort of triangle, i.e., all triangles are treated as the same in contrast to nontriangular figures. During Substage IIB (6-7 years) the enlargements begin to take account of parallelism between the pairs of sides, but only in a few special cases, such as when the enlargement is relatively small and the parallelism is apparent.

Stage III (7-11 years) marks the appearance of operations facilitating general comparison of parallels, angles and simple dimensional relations, such as the ratio of 1:2. At this level, the child is able to draw and compare similar triangles, but not similar rectangles. The latter requires some knowledge of dimensional proportions, whereas recognition of similarity of triangles depends only on elementary qualitative operations. At Stage IV, the child attains a concept of proportionality for all dimensional relations.

Development of Euclidean Space

Elementary topological relations are concerned with the object as a thing-in-itself and/or with various features taken successively. Projective concepts, in contrast, imply a comprehensive linking together of separate objects into a single system based on coordination of different viewpoints. Concurrently with the development of an organized complex of viewpoints, coordination of objects as such emerges. Ultimately this latter development provides for the transition to Euclidean space, with the concepts of parallels, angles and proportion providing the link between the two systems. Such a coordination of objects requires conservation of distance together with some notion of 'displacement' (or congruent transformation of spatial figures) which culminate in the construction of systems of reference or coordinates.

At the outset, the coordinates of Euclidean space appear to be no more than a network embracing all objects and consist of relations of order applied simultaneously to all three dimensions. Within this network, each object is linked simultaneously with the rest in three directions; left-right, above-below and before-behind, along straight lines parallel to each other along one dimension and intersecting those belonging to the other two dimensions at right angles.

A reference frame proper, however, is not simply a network composed of relations of order between the various objects. It applies equally to positions and enables the relations between objects to be maintained as invariant and independent of potential displacement of the objects. Thus, a frame of reference constitutes a Euclidean space after the fashion of a 'container'; and is relatively independent of the mobile objects contained therein, just as projective coordination includes the totality of potential viewpoints.

The simplest and most natural reference frame available to the child is probably that provided by the physical world in the shape of vertical and horizontal axes. On the empirical level, the horizontal is given by the plane on which everyday objects rest, the earth itself (where flat), or the artificial planes of floors, terraces, and so on. Another important factor is the surface of a liquid, e.g., surface of a lake or level of water in a glass. Walls of rooms and houses, posts, chimney stacks, trees, etc. provide experience with verticality. The study of the construction of horizontal and vertical axes provides a suitable way to understand the construction of coordinates in Euclidean space.

The task for studying horizontality uses two narrow-necked bottles: one with straight, parallel sides and the other with rounded sides. Each is partially filled with colored water and the children are asked to "guess" the position of the water when the bottle is tilted. The study of the vertical involves floating a cork with a matchstick placed vertically in it on the surface of the water in the jars. The child is asked to draw the position of the "mast" of this "ship" at different inclinations of the jar. In addition, the child is shown a mountain of sand and asked to plant posts "nice and straight" on the summit, on the ground nearby, and on the slopes of the mountain.

The results show that at Stage I (up to 4-5 years), the child is unable to represent either the water or the mountain as a plane surface. At Stage IIA the child realizes the water as a plane surface, but always parallel to the base of the bottle even when the bottle is tilted. It is surprising, but children at this level not only fail to note that the water level is always horizontal in their everyday observations but also do not use the results of the experiment when it is performed (i.e., with complete perceptual information available)!

During Stage IIB, the child no longer draws the water level parallel to the base of the jar when the jar is tilted, but he still fails to coordinate his predictions with any fixed reference system outside the jar (i.e., with the table or the stand). Also, children at this level are usually able to stand the posts upright when planting them in the sides of the sand hill, but continue to draw them perpendicular to the incline.

At Stage III (7-8 to 9 years), the child is able to predict the horizontal and vertical in all cases which requires a system of coordination or series of comparisons between objects in different positions and orientations. These comparisons are made by linking the various objects together in a system where stationary objects (the table) serve as reference points for mobile ones (the liquid surface).

The conclusion to be drawn is that Euclidean relations, completed by the construction of reference frames, are essentially relations established between numbers of objects and serve to locate them within an organized whole. Thus, horizontal-vertical axes are constructed at the same time as perspectives and are coordinated since these latter also constitute overall systems linking together objects or patterns. But projective space is in essence a coordination both of viewpoints, actual or virtual, and of the figures considered in relation to these viewpoints. Coordinates, on the other hand, link together objects, as such, in their objective positions, displacements, and relative distances. The age of 9 (approximately) is midway in the period during which concrete operations first take shape and thus marks a decisive turning point in the development of spatial concepts, i.e., completion of the framework appropriate to both Euclidean and projective systems.

Conclusions

Piaget maintains that the evolution of spatial relations proceeds at the two distinct levels: perceptual and operational (conceptual). Several years of experience separate the full development of perceptual and conceptual construction of space, but similar paths of development (topological, followed by projective and Euclidean) are observed.

Topological relations (proximity, separation, order, enclosure and continuity) are the most elementary spatial relationships both from the genetic-psychological and mathematical point of view. The topological relations, with which the child begins to construct his concept of space, are transformed concurrently into projective and Euclidean concepts. Projective space introduces a 'point of view', and Euclidean space introduces 'distance' and 'measurements" into topological space. The first of these, embracing perspective, section, projections, and plane rotations, results from the coordination of viewpoints, while the second derives from the conservation of straight lines, parallels, angles, and lastly, general coordinate systems.

In psychological terms, topological space relations consist of such elements as "A is *near*, *beside*, *far from*, *in(side)*, *out(side)* B," or "A is between B and C." Projective space introduces relations such as "A is *before/behind*, *above/below*, *or to the right of/to the left of B*," and Euclidean concepts add quantitative aspect to these projective relationships; Euclidean relations consist of the additional concepts such as "*How far* is A before/behind, above/below, or to the right/to the left of B."

Elementary topological relationships subsist between neighboring parts of a single object, or between an object and its immediate environment. Such a space is merely a continuous collection of elements which may be expanded or contracted; neither straight lines, distances, nor angles are conserved. Consequently, topological concepts do not lead to the construction of a stable system of figures, nor to fixed relations between such figures. Topologically, each continuous domain constitutes a space, and thus there is no universal space operating as a frame and enabling objects or figures to be located relative to one another. There are, for the child, as many spaces as there are objects or distinct patterns, the intervals between more distant elements either belonging to the elements themselves or not being spatial at all. In this connection, projective concepts perform a vital role in bringing about a global coordination of space.

Projective concepts take account, not only of internal topological relationships, but also of the shapes of figures, their relative positions and apparent distances, though always in relation to a specific point of view. Unlike the coordinate system implied in Euclidean space, a projective system does not conserve distances and dimensions, but does conserve the relative positions of parts of figures or of figures relative to one another and the whole in relation to the plane corresponding to the observer's visual field. From the psychological standpoint, the essential feature here is the inclusion of the observer (or a 'point-of-view') in relation to which the figures are projected. Elementary projective concepts are therefore based on the same operations as are topological concepts, but with the addition of a 'viewpoint'. The linking this 'viewpoint' with *operations of order* is basic for the construction of the projective straight line. A straight line is a series of points so arranged that from the 'end-on' viewpoint they are in alignment and are reduced projectively to a single point. Similarly, the notion of spatial dimensions can be defined more clearly in terms of certain sets of conditions specified by a given viewpoint. Topologically, the first dimension corresponds to a linear series, the second to the notion of inside and outside a closed linear boundary, and the third to the notion of inside and outside a closed two-dimensional boundary (surface). The addition of a perspective viewpoint to which the figures are related permits these same relationships to embrace relative orientation as expressed by 'on the left or on the right', 'above or below', and 'before or behind'.

Intermediate between the projective and Euclidean relationships, there arise certain relationships (affine and similarity) which children begin to understand at about the same time as they master the projective relationships. Projective relationships conserve neither parallels, angles, nor distances. Affine relationships, on the other hand, conserve parallels while angles and distances continue to vary. In the case of similarities, however, the figure retains its shape unchanged (straight or curved lines, parallels and angles) but changes in size according to relations of proportionality. Finally, Euclidean relationships add the notions of distances and measurement.

To conclude, topological and projective geometry are concerned with qualitative properties of space, whereas Euclidean geometry introduces quantitative properties to space.

Child's Conception of Geometry

The techniques Piaget (Piaget, Inhelder, & Szeminska, 1960) used to study conservation and measurement of length, area, and volume will first be described briefly. Second, some general findings will be presented. Finally, processes assumed to be involved in the child's construction of metric properties of space will be considered.

Methods

The results obtained from studies of children's spontaneous measurements and comparisons of two straight lines are described in some detail so as to clarify the general context in which measurement and conservation behaviors develop.

Spontaneous measurement was studied by showing a tower made of twelve blocks of cubes and parallelepipeds. The tower was 80 cm in height, and the child was asked to build a similar tower on another table. The instructions deliberately avoided mention of measurement; the experimenter used phrases such as: "You make a tower the same height as mine."

For conservation of length the child first was shown two straight sticks identical in length and with the ends aligned. One of the sticks was then moved forward 1 or 2 cm, and the child was asked which of the two was longer or whether they were the same length. In a second task, twelve to sixteen matches were arranged in two parallel rows and 1-2 cm apart. One of the rows was then modified by placing the matches at an angle.

The question to be answered was always whether the two lines were still the same length. Finally, two strips of paper each 30 cm long and about 1 cm wide were placed before the child, and he was asked to assure himself the two strips were identical in length. One of the strips was cut, first in two parts, then in several, and parts arranged in an arc, etc. The same questions as those in the first task were asked.

The technique for studying measurement of length was a direct extension of that used in the conservation of length task. The child was asked to judge between strips of paper in a variety of linear arrangements, involving right-angles, acute angles, etc., but these were now pasted on cardboard sheets. After his replies ("equal" or "one is longer"), he was shown several movable strips and asked to verify his judgment: "Have a look with this and see if you're right. Try and measure." Finally, he was given short strips of cardboard 3 cm, 6 cm, (these lengths corresponding with those of segments on the mounted strips) to aid in verifying his judgment. The experimenter also demonstrated by applying the 3 cm card two or three times along the mounted strips, beginning with the point of origin, and explained "a little man is walking along a road and these are the successive 'steps' he takes as he walks." In this latter case, the child was asked to finish the task as he was shown.

The conservation of area tasks were composed of several separate sections which permitted a modification of the arrangement of parts to test whether or not the child considered the whole to remain constant. For example, two cardboard rectangles, each made up of 6 squares, might be used. The twelve such squares were all equal, and each of the rectangles was two squares wide and three high. After constructing the rectangles, the experimenter transferred the top right-hand square on one to the bottom right-hand corner, which yielded a pyramid of three squares in the bottom row, two in the second, and one at the top. The child was asked whether this figure had the same area as the other rectangle which was left intact. In a second task the child was shown two rectangles, recognized as congruent, from which the experiemnter cut off a portion of four corners, putting them against the sides to produce an irregular polygon, etc. (any congruent figures can be used instead of rectangles if desired). The questions were always: "Are these the same size?" "Is there the same amount of room?" etc.

Measurement of areas was studied by using two tasks: (1) Measurement by superposition involved test objects consisting of a large right triangle (A), an irregular figure (B), and triangles (squares cut diagonally in half). There were sufficient smaller shapes to cover the whole of B and more than enough to cover A. The child was asked to use the smaller shapes to cover the large figures. (2) For measurement by unit iteration, the child was shown a number of shapes equal in area but differing markedly in shape. One (A) was a square that could be composed from the nine smaller squares, and the others (B and C) were irregular figures made up of the same number of small squares. The child was given a cardboard square representing one unit together with a pencil and was free to examine the material, and to draw on it. If he did not know what to do, the experimenter demonstrated how to use the unit-square to cover the large square if necessary. When finished with A, B, and C, the child was given two more shapes that were more heterogeneous and not equal (D and E). The child was offered a choice of three counters to use for measuring: a square, which was a quarter of D, a rectangle double the squares (so that two would fit into D), and a triangle equal to a square cut diagonally in half. The child was asked to use the small squares or triangles to fit into the large figures.

Conservation and measurement of volume was observed by showing the child a block measuring 4 cm in height with 3 cm x 3 cm base (volume = 36 cubic cm). The block was presented as "an old house" built on an island, (a square cardboard, 3 cm x 3 cm pasted on a sheet of corrugated card). The house is "threatened," so the inhabitants decide to build another which is to have exactly as much room. The child is shown these other islands which are also pieces of card but which differ from the first in size or shape or in both, their measurements being 2 x 2 cm, 2 x 3 cm, 1 x 2 cm, 1 x 1 cm, and 3 x 4 cm. The problem consists in reproducing the volume of the first block while altering its form to comply with the new base. The equal volume must be built from wooden cubes of 1 cm^3 (the original block is solid). Equality of volume was expressed by "as much room," with further explanation as necessary.

Results

The results are presented here in three sections: first, results from the study of children's spontaneous measurements; second, the results from the study of the comparison of two straight lines; and third, a summary of the results of studies of conservation and measurement of length, area, and volume.

Spontaneous measurement. The responses made by children in the study of spontaneous measurement are summarized in Table 2.

Table 2

Levels of Development in Childrens' Spontaneous Measurement

Stage 1	(4 - 4½ years)	Visual transfer
Stage II	(4½ - 7 years)	IIA: Manual transfer IIB: Body transfer
Stage III	(7 - 8½ years)	IIIA: Transitive congruence IIIB: Unit iteration

At Stage I, visual transfer is the only basis of comparison between the two objects (e.g., towers). Comparison of the heights of two towers is made entirely by moving the line of vision; the subject makes no effort to move one of the towers closer to the other.

Substage IIA is characterized by visual transfer being supplemented by manual transfer. The towers to be compared are now brought together so that an appraisal of "neighboring" objects is made. In Substage IIB, children use their bodies, e.g., the span of hands or arms to "transfer" the height of the tower from one to another (body transfer). Such behavior is considered the beginning of the use of a middle term, but the transitivity involved here is still intuitive.

During Stage III transitivity in the operational sense is understood. The smaller as well as the larger object (term) is used as the middle term. For example, the length required is noted; if the "rule" is too short, they stop and go back and forth between the towers. Eventually, any object available is used as a common measuring rule. Such an object is stopped as often as necessary, which is the equivalent of assigning a "unit-value" to a given length. This operation of unit iteration marks the appearance of a metric system.

Comparisons of lengths. The results of comparing two straight lines are summarized in Table 3.

Table 3

Levels of Development in Children's Comparisons of Two Straight Lines

Stage 1 & IIA	Nonconservation
Stage IIB	Intermediate responses
Stage III	Conservation

Stage I and IIB behavior is characterized by no conservation of length, i.e.,judgements depend exclusively on the perceptual characteristics of the setting. Thus, when the child visibly focuses on the leading extremity of a moving stick, that stick will be judged longer and the progressive changes at the other end of the stick are ignored.

The beginning of Stage IIB is observed when intermediate responses occur, i.e., the child's responses oscillate between nonconservation and conservation. For example, the child concentrates first on one end of the pair of sticks and judges the top stick to be longer (because it projects beyond the other), but the next moment he focuses on the other end-point where the lower stick projects beyond the top one and he now decides that the lower stick is longer.

Conservation of length defines the Stage III level and results from the operation of compensation. When the sticks are being staggered, the child often responds: "The sticks are still the same length; there's a little space here (difference between the leading extremeties) and there's the same little space (difference between the trailing extremeties)."

Summary

The development of conservation and measurement in length, area, and volume follows similar courses as those observed in the studies of straight line comparisons and spontaneous measurement. The only differences lie in: (1) Content, i.e., the number of dimensions involved. (2) The measurement of volume is relatively inadequate at level IIIB as compared to that of length and area at the same level. One cannot apply a unit-volume over the total-volume to be measured in most cases as is possible for length and area, i.e., often many elements are hidden from view. (3) Calculation of length based on linear units appears complete at level IIIB, but not so for area and volume. Again, the child can perform unit iteration at IIIB level but "multiplication" (e.g., for a rectangle, 2 cm x 3 cm = 6 cm) operations are not realized until Stage IV. Responses of the children in the experiments of conservation and measurement of length, area, and volume are summarized in Table 4.

Discussion

To measure requires some "thing" as a base unit and transposing of that unit in a systematic way to the whole of the object to be measured. Underlying the Piagetian concept of measurement are the concepts of "conservation of size," "subdivision," "change of position," and a "coordinate system." The measurement of space, thus, is not possible without the establishment of conservation of size, and of the coordination between change of position and subdivision.

The achievement of conservation of size depends on recognition of the distinction between empty space as "container" and solid moveable objects as "contained." If, for example, two straight sticks of equal length are first laid end to end and then slightly staggered in relation to one another, Stage I and II subjects say that the lengths are equal, but at Stage III they are convinced of the equality because now it is "recognized" that newly occupied "sites" compensate for places left empty by the change in position. This awareness of compensation is based on the discovery that properties of length, area, or volume remain invariant when position is changed. But, the discovery of these invariants in turn depends on knowledge that when an object undergoes a slight change of position the space left unoccupied by the change is

Table 4

Summary of Children's Responses in Conservation and Measurement

	Conservation	Measurement
Stage I & IIA	Nonconservation	Perceptual comparisons
Stage IIB	Intermediate Responses	Intuitive transitivity*
Stage IIIA	Conservation	Operational transitivity*
Stage IIIB		Metric measurement: unit iteration***
Stage IV**		Metric measurement: mathematical multiplication***

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*Transitivity is intuitive when the transferring of a middle term in comparison is limited only to some certain favorable situations (e.g., when objects to be compared are neighboring each other), and the comparison is still not free from percepptual influence and thus is still only approximate. An operational transitivity, on the other hand, is not restricted by the situational factors; it depends solely on logical inference, thus, if A=B, and B=C, then A=C.

**Stage IV applies only to the development in the measurement of area and volume.

***The operation of unit iteration is not equivalent to mathematical multiplication. The former consists in measuring the size of an object by moving a unit measure of the same dimensions stepwise over the total object (i.e., using unit-length to measure length, unit-area to measure area, unit-volume to measure volume). The latter allows one to measure area using linear units (e.g., 2 cm x 2 cm = $4cm^2$); or one can measure volume using linear units (2 cm x 2 cm x 2 cm = 8 cm^3), or linear and two-dimensional units (e.g., 4 cm^2 x 2 cm = 8 cm^3).

exactly equivalent to the newly occupied space. While the argument is circular--since compensation between spaces and newly occupied spaces depends on the invariance of area and volume despite change in position, and the latter depends on the awareness of compensation--it is this reasoning which enables the child to recognize the conservation of size when objects undergo changes of position.

Conservation of size is but a precondition for measurement. Measurement also depends on the coordination of subdivision and change of position. The operations of subdivision and change of position and their relations to measurement are as follows.

Operations of subdivision. Let us suppose a straight line C with a given length. By the operations of subdivision and composition the line may be broken into a number of contiguous parts and these in turn can be reunited to compose the original whole. Such parts may be represented by a length A which, together with another A' to which it is contiguous, yields a more inclusive part B ($= A + A'$), while B in turn is completed by a final part B', so that together they yield the whole line ($B + B' = C$). A child at Stage IIIA understands composition and can therefore deduce the following relations: $A + A' + B' = C$; $B - A' = A$; $C - B = B'$; etc. Here the quantification is derived only from part-whole relations and not from relations between one part and another: The subject is aware that $A < C$; $A' < C$; $B < C$; $B' < C$; $A < B$; etc., without needing to know the precise lengths of A, A' and B'. Between these elementary parts, A, A' and B , there can be no relation other than that of qualitative equivalence which is derived from their common membership of C.

Operations of change of position. In the linear series above, the initial order of its three elementary parts was $AA'B'$. Any change in their relative position is simply a matter of altering that order to $A'AB$ or $AB'A$ etc. Similarly, to change the position of the total line C amounts to an alteration in the order of C relative to a series of reference elements. On the other hand, any change in the order of its own parts will not affect the total length (C) because $A' + A + B' = B' + A + A' = A + A' + B' = C$. Likewise, the forward movement of the line C does not affect its length; i.e., the compensation is between newly filled spaces and vacated spaces. But, compensation also can be expressed in the language of qualitative subdivision: by moving forward, the stick C is increased by a new element C at its forward end; at the same time it loses a part C which it leaves behind. But since change of position is a change in the order of things, the new part C' and the old part of C'' are regarded as equivalent, i.e., $C' = C''$, and the child argues $C + C' - C'' = C$ just as $C + C' - C' = C$.

Thus, both subdivision and change of position are qualitative in character, and each taken alone is not sufficient to give rise to measurement. This fact is clearly seen in the study of children's spontaneous measurement. A change of position without subdivision evokes intuitive judgements based on movement. The most elementary form of this response is for the subject to run his finger along the two lines and make a motor comparison. A slight advance is noted when the child transfers a span or the width of two or three fingers. However, all these response types are limited because subdivision is approximate; there are no definite "marks" for guidance. Subdivision without change of position also cannot result in measurement. The most elementary form is exemplified by responses of a child when asked to judge the length of two outlines. He lays a strip of paper along one part only of a single outline. Later, he lays another strip along part of the second outline and passes judgement on the relations between the two, without comparing the two measuring strips. What he calls measuring is simply comparing two outlines by splitting each one separately into sections, without transferring these sections from one outline to the other or comparing them with each other. Thus, when subdivision without change of position occurs, two objects cannot be compared in terms of metric unit, and there is not true measurement.

Measurement begins when one part (A) belonging to a whole (C) is compared with the remaining parts of the same whole by change of position (either its own or that of a common measure, used transitively) so that A (or its equivalent) is superposed on these other parts. This implies subdivision and change of position are fused into one single operation and no longer simply complementary. The operations alluded to above involve the alternate use of subdivision and change of position and not the two together. Thus, subdivision antecedes change of position and is not its consequence; but change of position itself also is quite independent of subdivision. This fact is illustrated in the conservation of size. There is always an initial subdivision of the whole into parts, and the relative position of parts is then changed so the various parts take one another's positions. The operations do not involve any direct or indirect comparison between the several parts. But when one part (A) is applied to the remainder of the whole (C), the subdivision is not independently given, it is generated by the change of position effected either by A itself or by its transitive equivalent. Thus, A is moved stepwise along C-A, giving first A=A' (so that B=2A), then A=A'=B' (so that C=3A). In this case the subdivision cannot be dissociated from the corresponding changes of position. The subdivision depends wholly on change of position, but the reverse is also true. It is this synthesis of qualitative operations which gives rise to unit iteration and so constitutes measurement. By applying one section over and over until the whole has been covered completely, the whole is effectively reduced to a multiple of that section and the section becomes a unit. Since the unit can be subdivided in turn, using one of its fractional elements as a sub-unit, it follows that any size whatever can be compared with any other by means of whole and fractional units.

The notion of unit iteration to this point, however, is not yet completely developed in the metric notion of space. Areas are measured only in terms of units which are themselves areas, and volumes in terms of unit-volumes. It is not until later that the child learns to calculate areas and volumes by linear units. This ability to apply mathematical multiplication to the measurement of space evolves, according to Piaget, from the realization of the continuity of space. Thus, surface can be reduced to line when it is thought of as an infinite series of lines; volume can be reduced to surface when it is thought of as an infinite series of surfaces. It is with this achievement (mathematical multiplication) involving area and volume that the child has reached the final phase in the construction of Euclidean space.

To summarize, geometry is the science of space. The child's notion of space changes with development. At first, the child only is able to conceive of space in terms of such relationship as neighborhood, order, betweeness and closure. Later, he learns to construct space by a 'point of view' of the observer(s), and to describe space in terms of left-right, before-behind, and above-below. At the final stage, the child can conserve distance, and with the aid of a coordinate system, begins to conceive space in metric terms.

The three stages in the evolution of the child's notions of space correspond to spatial relationships constituting three branches of geometry. The conceptualizations that appear first in psychological development are the ones constituting the topological space; those appearing next constitute projective space; and those appearing last in psychological development are ones that constitute Euclidean space. The evolution of these notions of spatial relationships constituting topological and projective spaces has been treated in The Child's Conception of Space (Piaget & Inhelder, 1967). The development of the child's understanding of Euclidean space was left to a separate volume because it involves the complex notions of distance and measurement. The Child's Conception of Geometry (Piaget, Inhelder, & Szeminska, 1960) continues the work by treating these problems. In the latter volume, it was argued that conservation of distance is based on the distinction between movable objects and fixed sites as reference points. Measurement of space presupposes the notion of the conservation of distance and actually begins as the operations of subdivision and change of position are fused into a single operation.

Final Thoughts and Directions for Future Inquiry

A theory as comprehensive as Piaget's quite naturally draws much attention. Replications are attempted, discrepancies are found, and research based on the theory tends to, initially, be multi-directional. It would be surprising, indeed, if no findings contradictory to the theory were reported. What is surprising in this case, is that so much of Piaget's findings have withstood the empirical onslaught. Piaget's (1970) more recent distinction between figurative and operative thought gives the theory more flexibility for dealing with a broader range of psychological problems, including "real life activity." Youniss and Dennison (1971) have confirmed some of the implications of the figurative and operative distinctions. They have shown that the two processes are complementary within development levels but with operative thought dominating the figurative processes. Also, Piaget's theoretical structure too often has been considered without the role it gives to experience, which varies from one individual to another; the critical point is that modes of dealing with experience are similar among persons.

Piaget's biological tradition has often been contrasted with the psychometric tradition in the U. S., but the two approaches may not be so far apart. Laurendeau and Pinard's (1970) extensive study of five Piagetian tasks, which confirmed many of Piaget's findings using a great many subjects, made good use of scalogram analysis. The technique is gaining in popularity, since it supposedly answers the question of whether a set of behaviors occur in an orderly (developmental) sequence. Laurendeau and Pinard's research also sets an important precedent, that is, using the same children for many tasks within the same study. The analysis of interrelations among task performance of the same children is necessary to describe the common underlying cognitive structure. For example, Kaufman (1971) examined the factor structure of some of Piaget's and Gesell's tests, and the Lorge-Thorndike tests in 5-6 year-old children. The Piagetian tasks reflected three factors, corresponding to the operation of number, classes, and relations. Bart (1971) reported a general formal operational factor and a second factor related to task context. Berzonsky (1971) found five separate factors (in 30 variables with first-grade children). Shantz (1966) did not use factor analysis, but did study the interrelationship among tests of class multiplication of asymmetric logical relations and spatial multiplication in concrete operational children. The tasks were found to intercorrelate moderately, but exhibited a variety of more specific complex relationships.

A point often neglected is that a moderate correlation between scores on tests of multiplicative abilities, for example, is not to say that the relationship between multiplicative abilities is moderate. The latter statement could be made if all relevant variables are considered, but it is difficult to know when this has been achieved since adequate data are still not available. Researchers, understandably, wish to develop their own tasks, but contradictory results lead to question of the validity of the instruments. Even the validity of the Piagetian tasks (since primacy does not imply validity) is not known. It does seem, however, that too many new tests are developed prior to a thorough understanding of the theoretical imperatives for tasks that might provide a test of the theory. There is the nagging suspicion that some tests are developed with a hopeful eye toward being able to test a child and, with certainty, place him in Stage X. The dangers of assessing a child's cognitive structure(s) on the basis of a few "tests," especially using standardized psychometric methodology, was pointed out early by Inhelder, Bovet, Sinclair, and Smock (1966).

One further thought. Gagné (1968) would leave one devoid of any hope for Piaget's theory. "I believe that many of the principles mentioned by Piaget, including such things as reversibility. . . are abstractions. . . obviously in Piaget's mind. But they are not in the child's mind" (p. 188). Surely what Gagné means is that an (external) observer attaches a label to mental operations. It really is unlikely that Piaget thinks the child goes around saying, "I am practicing reversibility." When it comes to the brass tacks of a counter theory, Gagné can only offer that specific "euthenics" which are learned are generalized by combining with other "euthenics," "by means of a little understood, but nevertheless dependable, mechanism of learning transfer" (p. 189). Our only comment is that this phantom mechanism also has gone unnamed by all children I know.

What one does when two theoretical positions are in conflict is rarely a matter of empirical or logical compulsion. The learning and developmental approaches often speak of the same phenomenon, using different terms, leaving one searching for a third language which would make everyone happy. What one does about a great deal of the data on the concept of space, contradictory to or requiring comment by Piaget's theory, is also largely a matter of choice. Given our current state of knowledge, the theory, if one believes it, can be made to account for most all of the findings discussed here. Relevant systematic inquiry into Piaget's foundation has a relatively short history.

The direction of future inquiry is set by the critical concepts and methodological imperatives of Piaget's theory. Psychological research by and large has been, until about 1960, limited to psychological analysis of "stages" rather than "change" and based on response choice rather than actions and transformations. For example, most studies of development of concepts of space are concerned with what cues are perceptible, i.e., is the child able to discriminate between a form or object and its mirror image. The important question, from the Piagetian perspective, is whether the child can identify an object that is a 180° change from the standard object. More generally, can the child distinguish a change in position from a change in state? Thus, the strategies used by the child to distinguish various transformations constitutes the bases for describing developmental change. The main point of Piagetian theory is the shift from analysis of "features" to the coordination of input and action (Inhelder et al., 1966).

Each advance in science is accompanied by methodological and/or technical innovations made "realizable" by the new ideas. Piaget's contribution in this area is his refined observational procedures designed, as much as possible, to give the child opportunity for "spontaneous construction." Psychology has been a psychology of controlled choice and not a psychology of transformations. Piaget's epistemological imperative to psychology is that learning occurs by making transformations on objects and, further, that learning results from self-generated transformations:

> [In manipulating] besides learning something about the object in the course of such an experiment, the child also learns something of the way actions are coordinated and how one determines another. (Piaget & Inhelder, 1967, pp. 453-454)

The usual method of psychological research is to present successive exposure to stimuli controlled by the experimenter. The structuralist (Piagetian) approach, on the other hand, requires situations that permit self-regulation of patterns of action sequences. Of course, many psychological researchers believe that such procedures, while theoretically relevant, are too imprecise and not appropriate for the science of psychology. If, for example, input variables are determined by the "whim" of the subject, how can "causes" be identified? The answer depends on our definition of "explanation." If the concept of "cause" involves the issues of "antecedents-consequent" relations, then the structuralist approach appears naive and out-dated. However, if formal (as contrasted to efficient) causality is accepted as a type of "explanation," there is no problem.

Research involving the psychology of transformations, including those concerned with space-time, requires a much finer grain analysis than the analysis that is typical of past behavioral studies. The microanalysis of development a la Piaget is essential to further our knowledge generally and for testing Piaget's theory. Aside from the latter, microanalysis of action patterns under controlled conditions will fill in apparent discontinuities in cognitive development (i.e., to discover and explicitly describe the nature of the functional interactions that give rise to the hypothesized emergent functions).

As Forman (1973) suggests, microanalysis has the further advantage of placing the focus on the child rather than on the task. It is not sufficient to know the developmental order of a task; rather the experimenter must observe, record and analyze the process (action pattern) of task solution. Otherwise, the investigator can only make indirect interpretations, i.e., those derived from his reasoning of what is logically required to solve the particular problem. An example from Forman (1973) follows:

> It has been reasoned that the child can distinguish a two dimensional 'convexity' by using the convention that shadows come from above (Yonas, 1973). Another reader could enter with his 'yeah, but' and point to other features of the task which might have been the cues of response. If the researchers would take a closer look at the child, say with high-speed photography, they may find something in the nature of the research that would make the conclusions firm. What if the child extended a single finger toward the 'concavity,' as if to place his finger 'inside,' but when approaching the 'convexity' he maintained an infantile thumb-forefinger opposition as if to pinch at the 'protrusion'.

That microanalysis of behavior can add significantly to the understanding of spatial development is clear from the work of Forman (1973) and the study of visual scanning patterns by Vurpillot (1968), among others.

Emphasis on transformation capability, microanalysis of behavior (i.e., logic action patterns) and open-ended response conditions all require a shift in our basic paradigm about experimentation (Smock, 1973). The full meaning and implications of this Piagetian "revolution" are now becoming clear (Smock & von Glasersfeld, 1974).

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Breakthroughs in the Psychology of Learning and Teaching Geometry

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Russian mathematicians and mathematics educators have always been very fond of geometry. To some extent, this fascination is due to their pride in the achievements of Lobachevskii. Whatever the reasons, geometry has always played a central role in the Russian school mathematics curriculum. But even in Russia children traditionally began their study of geometry as a separate subject relatively late, namely in grade 6 (that is, at the age of 12 or 13).

This late start occurred largely because the development of the student's deductive and logical reasoning powers had been inappropriately considered to be a principal goal of school geometry instruction. And yet, the course constructed on the basis of this objective demanded of the students a high level of general development which they had not attained in their previous instruction.

In the Soviet Union and in the rest of Europe, the absence of this necessary development posed the problem of preparing students for the beginning of geometry study and, as a result, a preparatory course in geometry was developed -- a course sometimes called visual, concrete, intuitive, heuristic or propaedeutic geometry. Thus, two approaches to geometry have been used in Soviet Schools: the intuitive for grades 1 through 5 and the systematic (semideductive) beginning in the sixth grade.

Soviet educators who had been teaching either goemetry sequence experienced acute dissatisfaction with the conditions of knowledge prevailing in most of their students, as had their counterparts in the rest of the world. Extra hours of individual work and supplementary lessons with slow learners did not produce the desired results. The pupils committed errors again and again, showing their basic inability to solve the simplest problems on their own. An obvious question presented itself: Why was it that so many children who mastered most school subjects got nowhere in their study of geometry? Over the past thirty years Soviet mathematics educators and psychologists have been making a thorough analysis of the content and methods of teaching both the intuitive and systematic courses, and have tried to find answers to this question.

Excellent research indeed has been conducted at the USSR Academy of Pedagogical Sciences in order to improve the situation. Some of the Academy's work is included in Soviet Studies in the Psychology of Learning and Teaching Mathematics (Kilpatrick & Wirszup, 1969-1972; Kilpatrick, Wirszup, Begle, & Wilson, 1975), the series published jointly by the School Mathematics Study Group (SMSG) of Stanford University and the Survey of Recent East European Mathematical Literature of the University of Chicago. Volume 1 of this 14-Volume Series contains samples of the research by Zykova on the Learning of Geometric Concepts and studies by Galperin and Georgiev on The Formation of Elementary Mathematical Notions. Volume 4, entitled Problem Solving in Geometry, includes papers by Kabanova-Meller, Talyzina and Yakimanskaya. Volume 5, on the Development of Spatial Abilities, offers research by Chetverukhin. Interesting studies by Artemov (The Composition of Geometric Skills), and Mashbits (The Formation of Generalized Operations as a Method for Preparing Pupils to Solve Geometry Problems Independently) appear in Volume 13, and a paper by Tishin (Instructing Auxiliary School Pupils in Visual Geometry) is included in Volume 10.

Still, this very significant research has influenced the improvement in the teaching of geometry only slightly. The truly radical changes and far-reaching innovations in the new Soviet geometry curriculum have, in fact, been introduced thanks to Russian research inspired by two Western psychologists and educators.

First, the Russians have accepted as a fundamental principle the well-known and crucial discovery by Jean Piaget (and his co-worker, Bärbel Inhelder) contained in a paper by Piaget entitled Les structures mathématiques et les structures opératoires de l'intelligence (1955) and in the book La genèse des structures logiques élémentaries, by Piaget and Inhelder (1959). Piaget asserts that traditional geometry instruction begins too late and then takes up the concept of measurement right away, thus omitting the qualitative phase of transforming spatial operations into logical ones. This is to say that instruction is realized in a sequence corresponding to the historical development of geometry -- from the "geometry of measurements" to the "geometry of shape" -- from geometry of position to theoretical geometry. But the development of geometric operations in children actually proceeds in the opposite direction -- from the qualitative to the quantitative.

However, it is the life work and research by two Dutch mathematics educators which contains the most profound psychological and mathematical ideas, and has formed the basis for designing the new Soviet curriculum and methods of teaching geometry in the U.S.S.R. Unfortunately, this work has remained unnoticed in the United States, and probably would have been ignored in Western Europe as well, were it not for Professor Hans Freudenthal, the famous mathematician and educator.

P.M. van Hiele, a teacher at the Lycée of Bilthoven in Holland, is the author of a dissertation (1957) on intuition, particularly on the role of intuition in the teaching of geometry. His late wife, Dina van Hiele-Geldof, defended her doctoral thesis on didactics in geometry before the University of Utrecht also in 1957. That same year P.M. van Hiele delivered a talk at a mathematics education conference at Sèvres near Paris, and in 1959 he published it in a paper entitled La pensée de l'enfant et la géometrie (The Thought of the Child and Geometry). Here he discusses five levels of thought development in geometry.

Mathematics educators, methodologists and psychologists at the Soviet Academy of Pedagogical Sciences hastened to organize intensive research and experimentation on the levels of development outlined by van Hiele, and between 1960 and 1964 they verified the validity of his assertions and principles. We offer here the van Hiele levels as given in the more elaborate Russian post-experimental description (Pyshkalo, 1968; Stolyar, 1965).

Van Hiele Levels of Development in Geometry

Level I

This initial level is characterized by the perception of geometric figures in their totality as entities. Figures are judged according to their appearance. The pupils do not see the parts of the figure, nor do they perceive the relationships among components of the figure and among the figures themselves. They cannot even compare figures with common properties with one another. The children who reason at this level distinguish figures by their shape as a whole. They recognize, for example, a rectangle, a square, and other figures. They conceive of the rectangle, however, as completely different from the square. When a six-year old is shown what a rhombus, a rectangle, a square, and a parallelogram are, he is capable of reproducing these figures without error on a "geoboard of Gattégno," even in difficult arrangements.[1] The child can memorize the names of these figures relatively quickly, recognizing the figures by their shapes alone, but he does not recognize the square as a rhombus, or the rhombus as a parallelogram. To him, these figures are still completely distinct.

Level II

The pupil who has reached the second level begins to discern the components of the figures; he also establishes relationships among these components and relationships between individual figures. At this level, he is therefore able to make an analysis of the figures perceived. This takes place in the process (and with the help) of observations, measurements, drawings, and model-making. The properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognizing figures. At this stage, the figures act as the bearers of their properties, and the student recognizes the figures by their properties. That a figure is a rectangle means that it has four right angles, that the diagonals are equal, and that the opposite sides are equal. However, these properties are still not connected with one another. For example, the pupil notices that in both the rectangle and the parallelogram of general type the opposite sides are equal to one another, but he does not yet conclude that a rectangle is a parallelogram.

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[1] The van Hieles have used the "geoboard" in their research so that the child will not be hindered by difficulties resulting from drawing the figures.

Level III

Students who have reached this level of geometric development establish relations among the properties of a figure and among the figures themselves. At this level there occurs a logical ordering of the properties of a figure and of classes of figures. The pupil is now able to discern the possibility of one property following from another, and the role of definition is clarified. The logical connections among figures and properties of figures are established by definitions. However, at this level the student still does not grasp the meaning of deduction as a whole. The order of logical conclusion is established with the help of the textbook or the teacher. The child himself does not yet understand how it could be possible to modify this order, nor does he see the possibility of constructing the theory proceeding from different premises. He does not yet understand the role of axioms, and cannot yet see the logical connection of statements. At this level deductive methods appear in conjunction with experimentation, thus permitting other properties to be obtained by reasoning from some experimentally obtained properties. At the third level a square is already viewed as a rectangle and as a parallelogram.

Level IV

At the fourth level, the students grasp the significance of deduction as a means of constructing and developing all geometric theory. The transition to this level is assisted by the pupils' understanding of the role and the essence of axioms, definitions, and theorems; of the logical structure of a proof; and of the analysis of the logical relationships between concepts and statements.

The students can now see the various possibilities for developing a theory proceeding from various premises. For example, the pupil can now examine the whole system of properties and features of the parallelogram by using the textbook definition of a parallelogram: A parallelogram is a quadrilateral in which the opposite sides are parallel. But he can also construct another system based, say, on the following definition: A parallelogram is a quadrilateral, two opposite sides of which are equal and parallel.

Level V

This level of intellectual development in geometry corresponds to the modern (Hilbertian) standard of rigor. At this level, one attains an abstraction from the concrete nature of objects and from the concrete meaning of the relations connecting these objects. A person at this level develops a theory without making any concrete interpretation. Here geometry acquires a general character and broader applications. For example, several objects, phenomena or conditions serve as "points," and any set of "points" serves as a "figure," and so on.

The use of these levels permits us to isolate (and study) the essential aspects of the development of geometric thought from the large complex of interrelated factors characterizing the development of thinking in general.

Discontinuity of the Learning Process

The van Hieles (van Hiele & van Hiele-Geldof, 1958) noticed the discontinuity of the learning process:

> The discontinuities are . . . jumps in the learning curve, [and] these jumps reveal the presence of levels. The learning process has stopped; later on it will start itself once again. In the meantime, the pupil seems to have "matured." The teacher does not succeed in further explanation of the subject. He and . . . the other students who have reached the new level seem to speak a language which cannot be understood by the pupils who have not yet reached the new level. They might accept the explanation of the teacher, but the subject taught will not sink into their minds. The pupil himself feels helpless; perhaps he can imitate certain actions, but he has no view of his own activity until he has reached the new level. At that time the learning process will take on a more continuous character. Routines will be formed and an algorithmic skill will be acquired as the prerequisites to a new jump which may lead to a still higher level. (pp. 75-76)

These levels are inherent in the development of the thought processes. Van Hiele stated and Soviet research has shown that the passage from one level to another is not a spontaneous process concomitant with the student's biological growth and dependent only on his age. The development which leads to a higher geometric level proceeds basically under the influence of learning and therefore depends on the content and methods of instruction. However, no method not even a perfect one, allows the skipping of levels. The passage from one level to the next requires a certain amount of time; but various methods allow the regulation of this time period. It is also possible that certain teaching methods do not permit the attainment of the higher levels, so that the modes of thinking that would be used at these levels remain inaccessible to the student.

Freudenthal drew special attention to the work of his students and colleagues, the van Hieles, not only because he considered their work to be a truly revolutionary development in the teaching of geometry, but also because of the underlying pedagogical and didactic theory. In his opinion, their theory of levels of thought, as well as their masterful experimental courses in concrete geometry, are achievements of educational research which should be recommended to all who are interested in mathematics education. In his monumental work, Mathematics as an Educational Task (1973), Freudenthal writes about the van Hieles' theory of discontinuities in the learning process as follows:

> When the van Hieles started teaching they were just as unprepared as many other young teachers; nobody had told them how to do it. Of course they had passively undergone teaching, maybe even observed their teachers' performing, but this was not enough. As time went on, they had the opportunity to discuss their teaching with each other and with others. They subjected their own actions to reflection. They observed themselves when teaching, recalled what they had done, and analyzed it. Thinking is continued acting, indeed, but there are relative levels. At the higher level the acting of the lower ones becomes an object of analysis. This is what the van Hieles recognized as a remarkable feature of a learning process, namely of that [process] in which they [themselves] learned teaching. They transferred this feature to the learning process that was the goal of their teaching [that is] to the learning processes of pupils who were learning mathematics. There they discovered similar levels. To me this seems an important discovery. (p. 121)

This is how Freudenthal (1973) describes the reasoning of a child who is at the third level:

> If the child knows what a rhombus is, what a parallelogram is, he can visually discover properties of these shapes. There are a lot of them; during the class discussion the children count them up. In the parallelogram opposite sides are parallel and equal, opposite angles are equal, adjacent angles sum up to 180°, the diagonals bisect each other, the parallelogram has a center of symmetry, it can be divided into congruent triangles, and the plane can be paved with congruent parallelograms. This is a collection of visual properties which asks for organization. I explained earlier how deductivity starts at this point; it is not imposed but unfolds itself from its local germs. The properties of the parallelogram are connected with each other; one among them can become the source from which the

> others spring, so does a definition arise, and now it becomes clear why a square shall be a rhombus and a rhombus be a parallelogram. In this course the student learns to define, and he experiences that defining is more than describing, that it is a means of the deductive organization of the properties of an object. (p. 417)

After analyzing a typical lesson in a geometry class, P.M. van Hiele (1959) writes:

> The teacher reasons by means of a network of relations which he comprehends, but his students do not. On the basis of this network he presents the mathematical relations which the students end up manipulating out of habit. Or, rather, the student learns to apply -- out of habit -- these relations of whose source he is unaware and which he has never seen.
>
> Apparently everything is completely according to expectation: the students will eventually have at their disposal the same network as the teacher. The possession of a network of relations which is identical for all who make use of it and ideal for expressing reasoning — a network in which all of the relations are connected in a logical and deductive manner; is this not the proper end of the teaching of mathematics?
>
> Let us not be too optimistic. First, a network of relations composed in this way is not founded upon the sensory experience of the students. Although it is possible that the network of relations itself has inspired some experiences for the student, the mathematical experiences that the student has been able to have are based completely on the network imposed by the teacher. This network, imposed and not understood, forms the basis of his reasoning. A network of relations which is not founded on previous experience risks, as we all know, being forgotten in a short time.
>
> Thus, the network of relations is an autonomous construct: it has no connections with the other experiences of the child. This means precisely that the student knows only what he has been taught and what is linked to it deductively. He has not learned to establish the connections between the network of relations and the real sensory world. He will not know how to apply what he has learned to new situations.
>
> Finally, the child has learned to apply a network of relations which one has offered him ready-made: he has learned to apply them in certain situations specially

> designed for him, but he has not learned to construct such a network himself in a domain as yet unexplored. On the other hand, if as a result of our teaching the students should obtain the capacity to construct a deductive relational network in a new domain, we will have achieved an optimal mathematical training. (p. 200)

The following points made by van Hiele (1959, pp. 201-203) may contribute to a more precise understanding of the levels of thought:

> A) At each level there appears in an extrinsic manner what was intrinsic on the preceding level. At the first level the figures were in fact just as determined by their properties, but one who is thinking at this level is not conscious of these properties.
>
> B) Each level has its own language, its own set of symbols and its own network of relations uniting these symbols. The transition from one level to the next is related to the broadening of language -- the appearance of new geometric and logical terms, definitions, and symbols. A relation which is "exact" on one level can be revealed to be "inexact" on another. Think, for example, of the relation between the rectangle and the square. Numerous linguistic symbols appear on two successive levels; they establish, moreover, the connection between the different levels and assume the continuity of thought in this discontinuous domain. But their meaning is different: It is shown by other relations among these symbols.
>
> C) Two people who are reasoning on two different levels cannot understand one another. This is what often happens with teacher and student. Neither of them succeeds in grasping the progress of the other's thought, and their discussion can be continued only when the teacher tries to get an idea of the pupil's thought process and to conform himself to it. Certain teachers give an explanation at their own level, inviting the students to answer questions. This is, in fact, a monologue, for the teacher is led to consider all the answers which do not belong to his level of relations as silly or beside the point. True dialogue must be established on the pupil's level. In this case, the teacher must often, after class, question himself about his students' meanings and strive to understand them.
>
> D) The maturation process which leads to a higher level unfolds in a characteristic way; one can distinguish several phases. (This maturation must be considered principally as a process of apprenticeship and not as a ripening

on the biological order). It is then possible and desirable for the teacher to encourage and hasten it. It is the goal of didactics to ask how these phases are traversed and how to furnish effective help to the student.

The phases which in the course of apprenticeship lead to a higher level of thought are, according to van Hiele (1959), as follows:

> Information: The student learns to recognize the field of investigation by means of the material which is presented to him. This material causes him to discover a certain structure. One could say that the basis of human knowledge consists in this: Man appears in a position to uncover a structure in all material, no matter how disordered it is, and this structure is perceived in the same way by many people -- as a result of the conversation on this subject in which they can engage.
>
> In the second phase, that of directed orientation, the student explores the field of investigation by means of the material. He knows then in what direction the study is geared; the matter is chosen in such a way that the characteristic structures progressively appear to him.
>
> Explanation takes place in the course of the third phase. The acquired experiences are linked to exact linguistic symbols, and the students learn to express themselves in the course of discussions about these structures which take place in class. The teacher sees to it that the customary terms are employed in the discussions. It is during the course of this third phase that the network of relations is partially formed.
>
> The fourth phase is that of free orientation. The field of investigation is in large part known, but the student must still rapidly find his way around this field of investigation. This is achieved by assigning tasks which can be carried out in different ways. All sorts of signals are placed in the field of investigation. They show the way to follow in order to reach the symbols.
>
> The fifth phase is that of integration: The student has been oriented, but he must still acquire an overview of the methods which are at his disposal. He then tries to condense into a whole the domain which his thought has explored. At this moment the teacher can encourage this work by providing global insights, but it is important that these insights bring nothing new to the student: They ought only to be a summation of what he already knows.

> As a result of this fifth phase, the new level of thought is reached. The student arranges a network of relations which connect with the totality of the domain explored. This new domain of thought, which has acquired its own intuition, has been substituted for the earlier domain of thought which possessed an entirely different intuition.
>
> The objectivity of mathematics rests on the fact that new symbols and networks of relations are understood in the same way by a number of different people. If one were to determine as the end of education the oneness of the relational network, one might confine himself to having this network assimilated. The student would then seem to understand the reasoning process perfectly, for he would come up with exact conclusions by taking this relational network as his base. But that would not mean that he would attach the same meaning to it as his questioner. This meaning cannot be extracted only from the language used; it depends on the experiences which have led to the formation of the relational network; that is, it depends on what has taken place on the lower levels of thought.
>
> If one does not take the content of the symbols into consideration, but only their relations, one can say that, from the mathematical point of view, everything is perfect. The student is capable of manipulating the relational network of deduction without error. But from the pedagogical or didactic point of view, and from the social point of view, one has wronged the student.
>
> The pedagogical fault lies in the fact that the student is deprived of the opportunity to become aware of his creative ability. From the didactic point of view, the student is prevented from discovering how new domains of thought are explored. Society has been wronged because the teacher has placed in the student's hands a tool which he can manipulate only in the specific cases which he has studied. (van Hiele, 1959, pp. 201-203)

The Soviet Geometry Curriculum

Need for Changes

As mentioned above, since 1960 the Russians have been conducting experimental studies of their students' levels of development in the study of geometry. In order to design a new geometry curriculum, they determined to what extent the van Hiele levels reflect the actual process of the pupils' development and to what extent the traditional system and teaching methods have helped or hindered their development in geometry. The following represents only a few of the many interesting conclusions contained in lengthy Soviet reports on this research (Pyshkalo, 1968; Stolyar, 1965).

Under the old curriculum only 10 to 15 percent of the students who finished fifth grade reached the second level. This delay appeared to be even greater in relation to the pupils' familiarity with geometric solids. An appreciable leap was not noted here until the seventh grade.

At the same time, it was established that the familiarity of an experimental class of second-graders with the geometry of solids enabled them to reach the indispensable second level and even to surpass, in this respect, the level of development of seventh-graders in the traditional school. Table 1 records the results of assignments on classifying solids (cylinders, cones, pyramids, prisms). In the last column are data on the performance of the assignments by pupils in the experimental second grade. These classes spent one month on a study of solids and a description of the shapes of objects, in the form of independent study.

Table 1

Percent of Success on Classifying Solids by Students in Traditional and Experimental Sections

Assignments	Grades							
	I	II	III	IV	V	VI	VII	Experimental II
Sorted out all solids correctly	--	--	1	4	3	19	50	75
Isolated one group of solids correctly	49	54	50	52	62	100	100	100
Correctly named each group of solids	--	--	--	2	1	3	37	49
Correctly named one group of solids	1	3	3	20	40	69	100	100

As a result of various experiments and analyses, Soviet educators have termed their traditional instruction in grades 1 through 5 a "prolonged period of geometric inactivity." Here one can observe a violation of the most important conditions for the development of any kind of thinking -- continuity of study and versatility. The following is a striking illustration of the situation under the old curriculum: During the first five years of instruction, the pupils became familiar with 12 - 15 geometric objects (the names of the figures and their elements; terms designating relationships and properties; apparatuses and instruments, etc.). On the other hand, in the first topic ("Basic Concepts") alone of the grade 6 geometry course, to which only 16 class hours were allotted, the pupils were required to assimilate nearly 100 new terms, including the names of figures and their parts (approximately 60), terms designating relations and properties (approximately 20), and the names of apparatuses, instruments, and their parts (approximately 20).

A detailed analysis of the standard textbooks in mathematics for grades 1 through 5 revealed the absence of any systematic choice of geometric material, large gaps in its study, and a markedly late and one-sided acquaintance with many of the most important geometric objects.

The investigations showed jumps across levels (primarily from I to III) with respect to a significant majority of the concepts studied and marked gaps. In addition, the study of geometric concepts encountered in each of the first five years of instruction continued at Level I, and then only from the quantitative aspect, like measuring length and area. It was evident that in the traditional geometry course for grades 1 through 5, preference was given only to those concepts that could be measured.

Of great value as a quantitative decription of the beginning period in the study of intuitive geometry is the number of exercises in the standard mathematics problem-books in which actual geometric objects are examined. Column III, in Table 2, lists the number of problems containing geometric terms. The problems referred to in the table as "effective" are important in that their solution is related to the pupils' geometric development. Their number is insignificant indeed and amounts to approximately 1 percent of all problems.

Table 2

Exercises in the Standard Mathematics Problem-Books

Grade	Total Exercises	Problems containing geometric terms		Effective geometry problems among them	
		Number	Percent	Number	Percent
I	929	27	2.9	12	1.3
II	1181	40	3.4	10	0.8
III	1300	102	7.8	25	2.0
IV	1142	160	14.0	15	1.3
V	1157	53	4.6	6	0.5
Total	5709	382	6.7	68	1.2

The studies demonstrated that in the sixth grade, beginning with the very first lessons, the teachers were required to do work corresponding to the first three levels of geometric development simultaneously:

1. To familiarize the pupils with geometric figures in order that they recognize them according to their shapes (Level I).

2. To study the properties of figures in a practical way and enable the students to recognize figures according to their properties (Level II).

3. To proceed with the main task of grade 6; that is, to order the properties that had been discovered experimentally and to give these properties a meaning. The students now had to formulate definitions, and should already have been able to connect properties and to logically derive some properties from others (Level III).

Obviously, this was an impossible task.

Table 3 illustrates the change in the number of geometric concepts studied in the experimental as opposed to the traditional courses. It should be noted that this marked increase occurred not so much because of an increase in the number of geometrical figures that were studied and used for measuring purposes, but because the relationship of geometric figures and their properties was systematically studied in the experimental courses. Table 4 gives the number of problems with geometric content in the new geometry program, and Table 5 gives a comparison between the traditional and experimental courses in this respect.

Table 3

A Comparison of the Number
of Geometric Concepts Studied
in the Traditional and the
Experimental Programs

Grades / Courses	I	II	III	IV	V
Traditional course	7	4	13	23	37
Experimental course	24	51	84	100	--

Table 4

Exercises in the Experimental Program

Grade	Total Exercises	Problems Containing Geometric terms		Effective Problems	
		Number	Percent	Number	Percent
I	1100	200	18	180	16
II	1200	230	19	200	17
III	1300	260	20	240	18.5
IV	1300	300	23	290	22
	4900	990	20	910	18.5

Table 5

Effective Problems with Geometric Content

Grades	I	II	III	IV
Traditional	1.3%	0.8%	2%	1.3%
Experimental	16%	17%	18%	22%

A New Geometry Course

The various Russian investigations led to the conclusion that radical and qualitative changes in character, structure, and direction were needed to build a new geometry curriculum. P.M. van Hiele (1959) suggested the following elaboration of a geometry course:

> The first part of a geometry course must insure the attainment of the second level of thought, which we shall call the aspect of geometry. The aim of the instruction is as follows: Geometric figures must become the bearer of their properties.
>
> One uses a set of concrete geometric shapes and materials the manipulation of which will lead the students to work out geometric figures on their own. The operations which the students carry out with this material will become

the base of a new relational network.

The second part of this course must insure the attainment of the third level of thought, which we shall call the essence of geometry or the aspect of mathematics. Here the aim of instruction is to absorb the relations which link properties of figures; for example, that the sum of the angles of a triangle equals 180°; that the alternate-interior angles formed by two parallel straight lines and a secant are equal. In addition, one begins during this period to logically order the properties of figures. The first property cited above becomes the antecedent of a new property: that the sum of the degree measures of the angles of a convex quadrilateral is 360°.

The materials used can include a series of congruent triangles or quadrilaterals with which the students attempt to construct a "pavement" or covering of some part of the plane. Here again the students discover an underlying structure through the manipulation of concrete materials. They see systems of parallel straight lines, parallelograms, trapezoids and hexagons appear with centers of symmetry in the pavement constructed by means of congruent triangles. This material later furnishes a natural construction of the straight line which permits the demonstration by means of alternate interior angles that the sum of the angles of a triangle is 180°.

The third part of the course must insure the attainment of the fourth level, that of discernment in geometry, of the essence of mathematics.

The purpose of this instruction is to understand what the expression "logical ordering" means: For example, what is meant by some property preceding another.

The material is made up of the theorems of geometry themselves. Underlying the ordering of these theorems will be links between theorems and their converses and inverses, reasons why certain axioms and definitions are indispensible, and clear reasons for the necessity and sufficiency of certain conditions. The students will now be able to logically order new concepts. For example, when they study the cylinder for the first time, their analysis of what they perceive will teach them that the cylindrical surface contains straight lines and circles. After formulating a precise definition, they will be able to demonstrate the existence of straight lines and circumferences.

If the course could be pursued further (though that would generally be impossible in secondary education), it would reach the fifth level, that of *discernment in mathematics*. At this level, the goal of instruction would be to analyze what the mathematician's activity consists of and how it differs from that of other scholars. One can attain this fifth level only when he is so familiar with the steps of the mathematician that he executes them automatically; that is, he has established habits which force one step to inevitably follow another. Only then is an integration of these steps possible, allowing the person to grasp the structure of the activity in mathematics.

But a parallel integration has already been produced at the time of the transition from one level of thought to a higher level. In the course of passing from Level I to Level II, it is the manipulation of figures which brings the structure into the light. Manipulation nourishes thought on the second level. Thus, the figures become new symbols defined by their relations with other symbols.

On the second level the context differs from that of the first; the process undergone in this new context provides an integration which makes access to the second level possible, and so on.

The teacher who deliberately leads his students from one level to another allows them to develop a deductive system by themselves and to discover the faults in deductive reasoning. By acting thus, the teacher does not impose the domains in which the thought will be exercised, but helps his students to specify these domains themselves. This does not mean, as has been pointed out above, that he will leave the student the burden of discovering everything, but he will demand of him a particular activity which, in each of its five phases, is differently directed. An application of the principles presented here certainly does not imply a lightening of the teacher's task. It does, however, carry the satisfaction of knowing what one is doing and of better understanding the reactions of the students.

The teaching of a deductive system demands patience above all. Such a system exists only at the fourth level of thought; but its essence is perceived only at the fifth level. (pp. 204-205)

Here is how van Hiele views the axiomatic method in teaching geometry, from the point of view of his levels of thought:

> One makes a serious mistake in trying to construct a system of axioms by using symbols characteristic of a level of thought which is too low. Systems of axioms are in the province of the fifth level, where the question of what points, lines, surfaces, etc. are, is no longer asked. At this level the figures are defined exclusively by symbols connected by relations. To find their proper interpretations it would be necessary to return to the lower levels where the content of these symbols can be perceived. But with such concrete interpretations, these symbols belong to a relational network which cannot be axiomatized because it cannot have direct connections with logic. (1959, p. 203)

Soviet research (Pyshkalo, 1968) has concentrated on the van Hiele scheme for constructing a geometry course and on implementing the necessary drastic modifications of the forms and methods of instruction. This was accomplished over a period of years in a most organized and painstaking way, and led to the introduction of the new Soviet geometry curriculum.

One fundamental question was that of what should be studied in school: geometry or deductive systems with geometry as an example? Evidence was convincing that there was no basis for making the logical demands of the school course in geometry (as to a deductive system) any higher than those of arithmetic, algebra, grammar, and other subjects.

The research concluded that the most important factor in the improvement of curricula and teaching methods lies in establishing a single sequence in the formation of mathematical concepts for the entire eight-year school, beginning with the first grade.

Investigations indicated that the students in experimental grades 1 through 4 were able to develop a firm understanding of geometry without deductive formalization in its exposition, and that this period could serve as the beginning of a study of the semideductive (systematic) course.

On the other hand, the Russians claim that the period of accumulating facts inductively should not be extended too long. One may and must use deduction. There must be a timely introduction of a theory around which to unite the accumulated facts. In the new curriculum, elementary deductive conclusions are systematically reached by the pupils.

It was found that a marked economy in the further study of geometry could be achieved by (and on the basis of) the study of geometric transformations -- for example, of axial symmetry -- at an earlier time. (This would be the first topic for grade 5 -- 20 hours). In addition, the opinion was advanced that the study of geometry in an algebraic framework (the method of coordinates, vectors, elements of analytic geometry) could be of great value in the mathematics course.

In order to activate geometric development, research was conducted on methods of independent study, primarily with the use of workbooks based on the printed word, and carefully devised visual aids such as collections of models, tables and posters, films and slides. Special attention was given to these didactic materials from the point of view of individualized instruction. Excellent research was done in this area by a team headed by V.G. Boltyanskii, a well-known topologist.

The great bulk of independent work in mathematics in school has been intended as a kind of training in, or control of, the knowledge previously acquired by the pupils. In view of this, the students acquire hardly any skills in the independent study of new materials. It is well known that the role of the book as a source of information is rapidly expanding. Therefore, one of the principal tasks of Soviet instruction is to accustom the pupil to reading scientific literature on his own. However, one cannot read even the most interesting mathematics book as if it were a captivating story. Because of the laconic language and the high level of abstraction of mathematical concepts, the reading should be accompanied by drawing, construction of models, calculations, reproductions of proofs and conclusions.

It was verified that special films are an effective means not only of developing geometric concepts but also of forming spatial conceptions. Movies enable the student to analyze more complicated shapes, to establish the elementary ones, and to synthesize the more complicated ones from the individual solids.

Experimentation has shown that in grades 1 through 4 it is inadvisable to isolate the study of geometry material from the entire system working towards the students' mathematical education. In particular, in grades 1 and 2 one should not set aside special lessons for the study of geometry. In grades 3 and 4, the need does arise for the organization of separate lessons, entirely devoted to the study of geometry material.

Soviet investigations have also concentrated on devising special methods of teaching the new geometry course. The studies were aimed at finding detailed methods that would secure the course's basic content to be presented at the appropriate level of geometric thought development: Level I in grade 1, Level II in grades 2 and 3.

Listed among the chief goals in the study of geometry in the Soviet Union were:

Forming geometric notions

Developing thought (inductive and deductive thinking, analysis and synthesis, comparison, abstraction and generalization)

Forming spatial notions and imagination

Securing a connection between the study of geometry and other branches of elementary mathematics (arithmetic, algebra)

Developing skills

Using the visual principle (concrete object, model, drawing)

Methodological approaches were defined for every level of geometric development and involved taking the students' individual traits and potential into consideration.

Among the necessary modifications in the geometry curriculum and the principles underlying them were: the determination that the students' familiarization with geometric objects should begin with qualitative geometric operations (the study of shape, mutual position, relations, etc.) and that quantitative operations (measurements) should only be gradually developed somewhat later; the conviction that geometric studies should be *continuous*, allowing no gaps or periods of inactivity -- they should be *uniform*, allowing no overloading at any of the stages, and they should be *diversified*, treating many aspects of the study of spatial relations. This diversity should be also understood in the sense of a simultaneous familiarization of the students with two- and three-dimensional geometry. Since the student does not master isolated facts taken separately, but, instead, masters a system of interrelated facts; it is necessary to teach at each stage the interrelations among these facts and to contribute to the mastery of general principles. It is of great importance here to insure an organic connection between geometric objects and basic arithmetical concepts (i.e., to use the one to illustrate the properties of the other).

Special research was conducted to determine criteria for selection of geometry material for grades 1 through 3, especially from the standpoint of attaining an appropriate level of development. The criteria may be summarized as follows:

1. In grades 1 through 3 the students should be deliberately familiarized with most of the geometric concepts they will study in the eight-year school. These studies should first be qualitative and thus not limited to measurements.

2. The material for grades 1 through 3 should form a complete entity and play an independent role, insuring the students' formation of spatial conceptions and spatial imagination.

3. One must not only be concerned with the accumulation of a stock of geometric concepts and skills, but also with the attainment of an appropriate logical development, with achievement by all students of the second level of geometric development by the end of grade 3, and with mastery of the necessary geometric and logical terminology.

4. One must proceed from the fact that the students possess a significant store of conceptions of the properties of material objects (*geometry as physics*). Abstraction from some properties of material objects

allows clarification of the general quality basic to geometric concepts.

5. The point of departure of the geometry material must consist in relying upon a stock of geometric terms, in using correct terminology, and in working to discover their proper geometric content.

6. The curriculum should provide work in determining the shape of objects in the environment, based on a previously created stock of geometric concepts.

7. The study of relations between figures and of relative positions of figures should be undertaken.

8. It is essential to take into account the requirements of the disciplines studied concurrently in grades 1 through 3 and to include questions which apply to the study of these disciplines.

9. In planning work in the formation of measuring skills on an applied level, one should be concerned especially with forming conceptions of geometric quantities, and should use these skills and concepts in forming conceptions of number, of the properties of operations, and operations on numbers; this work should be correlated with the study of figures.

In the experimental geometry curriculum for grades 1 through 3 the following general plan was devised: an initial acquaintance with figures (grades 1-2), the study of properties of the figures (grades 2-3), the study of relations among figures (grades 2-3), measurement of geometric magnitudes (grades 1 through 3), the use of geometric figures and measurements in the study of numbers and operations on numbers (grades 1-3), and an introduction to concepts of set theory (grade 3). A model vocabulary and list of skills for each student to master were set up for each grade. The study of geometry was allotted 25 hours in grade 1, 30 hours in grade 2, and 40 hours in grade 3.

In 1963 the Sector for Mathematics Instruction of the Institute for General and Polytechnical Education (of the USSR Academy of Pedagogical Sciences) began experimental instruction in which a semideductive course in arithmetic was given in the fourth grade. It was established that at least the beginning of a semideductive geometry course could also be introduced in at least the fourth or fifth grade instead of the sixth grade (as in the old curriculum).

The Russian studies clearly show the advantages of a practical direction in instruction and the need for an emphasis on the close relationship between practical life at each stage of the curriculum. They have thus envisioned a wide range of skills in school work, each of which would contribute to more rational progress in the students' studies and could constitute a reliable means for the pupils to teach themselves, thus creating a quantity of primary concepts which are indispensable for understanding phenomena vital to the development of their thought.

The Russians have concerned themselves with developing skills in using various drawing and measuring instruments (in addition to the straightedge), as well as skills in modelling and constructing geometric figures and forming notions of their accuracy.

Systems of practical tasks (laboratory work) were devised at each stage of instruction. This material provides for the study of geometric objects not only in mathematics lessons, but also in other school subjects (such as drawing, manual work and athletics) connected with the pupils' academic and practical activity.

A system of exercises was worked out in conformity with the aims of the curriculum. As the experimental instruction showed, a need for an increase in the number of exercises usually arises as difficulties develop in learning new skills (such as skill in using instruments), in forming mathematical discourse, and in mastering special mathematical phraseology. These difficulties are surmountable when a sufficient number of specially-prepared exercises are done.

While space does not permit us to even sketch the extent of the Soviet research, or the quantity of their valuable results, we can say that in their experimental curriculum all pupils approaching the end of third grade completed work corresponding to the second level of thought development in geometry. This made it possible to begin studying semideductive geometry (at Level III) in grade 4, a course of study roughly corresponding to the first half of their traditional curriculum for grade 6. Also, enough evidence was accumulated to assume that students in the eight-year school are capable of reaching the same level of geometric development as has been attained in the traditional eleven-year school.

The experiments, research and studies mentioned above have had a decisive influence on the design and form of the new Soviet geometry curriculum, which has been introduced gradually since 1969. The USSR Academy of Pedagogical Sciences provided the geometry curriculum for grades 1 through 3; while the world-famous A.N. Kolmogorov, in close cooperation with such outstanding mathematicians as I.M. Yaglom, V.G. Boltyanskii and others, was most actively involved in curriculum design, text preparation, and formation of teaching methods and materials for grades 4 through 10. Again, space permits us to mention only some of the most striking features of the new curriculum:

The geometry course is given in three stages, and as a separate subject, starting at grade 4 and continuing for the next seven years. In all stages emphasis is placed on geometric transformations, and in the upper grades the course is based on vector representations. Beginning in grade 4, geometry is taught by a specialized mathematics teacher. The new curriculum is clearly the most radical change in Russian mathematics education in nearly a century.

As a result of unsuccessful experience and convincing evidence, the so-called axiomatic methods of initiation into geometry have been recognized by modern educators the world over as unpedagogical. A review of the teaching of geometry in the United States indicates at once that only a very small number of the elementary schools offer any organized studies in visual geometry, and where they are done, they begin with measurements and other concepts which correspond to Levels II and III of thought development in geometry. Since Level I is passed over, the material that is taught even in these schools does not promote any deeper understanding and is soon completely forgotten. Then, in the 10th grade, 15 and 16 year old youngsters are confronted with geometry for almost the first time in their lives. The whole unknown and complex world of plane and space is given to them in a passive axiomatic or pseudo-axiomatic treatment. The majority of our high school students are at the first level of development in geometry, while the course they take demands the fourth level of thought. It is no wonder that high school graduates have hardly any knowledge of geometry, and that this irreparable deficiency haunts them continually later on.

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Recent Research on the Child's Conception of Space and Geometry in Geneva: Research Work on Spatial Concepts at the International Center for Genetic Epistemology*

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About a decade after the original work of Piaget and Inhelder on the development of spatial notions was published (Piaget & Inhelder, 1948; Piaget, Inhelder, & Szeminska, 1948), Jean Piaget once again took up the study of space within the research carried out at the International Centre for Genetic Epistemology, hereafter called the Centre. The research then carried out had three interesting aspects. First, it naturally threw light on epistemological problems, since indeed the research was aimed at studying the nature of knowledge in space and covered in particular the following problems: Where does such knowledge take root? What is the role of mental imagery or physical experience in the development of space? What is the relationship between geometrical knowledge and other spheres of cognitive activity? Second, whereas epistemologists were concerned with the concepts developing in children and their relationship with the formal science of geometry, psychologists on the other hand can find in this research work varied and interesting information on the psychological processes involved when a child endeavors to surmount cognitive difficulties of a geometric nature. Last, to the extent that education should be based on a deep knowledge of psychological processes, the research work of the Centre can contribute substantially to improving teaching of spatial concepts.

The particularities of the problems studied will be summarized at a later juncture in this paper. But since they are based on a number of Piagetian theoretical concepts, it is deemed useful to refer briefly to them now.

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First, a distinction must be drawn between the figurative and operative aspects of knowledge. The function of the figurative aspect is to furnish an approximate imitation or copy of reality. Under this heading fall perception, imitation, and mental imagery. The latter, which Piaget considers to be an internalized imitation, has thus an active component, but it is basically suited to symbolize static aspects of reality. The function of the operative aspect of knowledge is to transform reality, and this transformational character is specific of intelligence. The elementary operative forms are concrete actions, and the advanced forms are mental operations which develop into systems that can be analyzed in terms of the "grouping" or group-like structures.

Two kinds of operations appear in children's thinking after the age of seven. First, logico-mathematical operations deal with discrete elements and consist, for instance, in gathering elements into classes and including subclasses into a total class, in relating the differences between elements, in adding or subtracting the elements, etc. Second, spatial operations deal with continuous objects (spatial figures or space considered as a whole) that they partition and then reconstruct. The two kinds of operations are analogous but not identical. For example, to the logico-mathematical operations involved in class inclusion correspond operations of partition which divide the whole into parts and establish proximity relationships between parts. To the logico-mathematical seriation of relations correspond operations of spatial ordering, i.e., direct order, inverse order and conservation of the position "in between."

Another distinction, which permeates all Piaget's work, must be made between the logico-mathematical and physical poles of knowledge. At the one pole, the logico-mathematical structures are drawn by reflexive abstraction from the subject's actions and allow for inference and ultimately deduction. They constitute structures which do not exist in the objects. For example, the number or the class of objects in a collection is introduced by the subject who counts or classifies them, but they are not attributes of the objects. At the opposite pole, the causal structures, or physical knowledge, are in part drawn from empirical observations in the child's endeavor to explain reality. These structures give account of structures of the outer world, such as spatial properties of objects or relations between physical variables (e.g., relation of speed, temporal order of starts and stops and distance covered). It must be stressed that physical knowledge does not stem from mere empirical observations, but also involves reflexive abstraction. Reflexive abstraction consists in abstracting something out of the organization of the subject's own actions (e.g., the rule of commutativity) and in rearranging the elements or rules abstracted on a new level (e.g., the level of representation opposed to the level of actions). The construct of reflexive abstraction plays two roles in Piaget's theory. First, it implies, contrary to empiricism, that something new can be learned which is not directly abstracted from outer reality, i.e., from the properties of objects, the laws to which objects are submitted, the ideas or models of actions of other individuals. Second, the construct of reflexive abstraction contributes to explanations of the continuity of development, from the biological organization of the individual up to the higher forms of intelligence. In effect, at each level of organization, something is kept from the level immediately inferior and is organized into new structures. Voluntary actions, for instance, are self-regulated like the activity of the nervous system, but they are progressively combined into more and more complex structures. At a level superior to the one of concrete actions, intelligence consists in combination of actions, but these actions are internalized and progressively organized into structures which are more complex and better "equilibrated" than the structures of actions. The continuity of intellectual development implies that complex scientific concepts such as those of geometry partly derive from the elementary spatial concepts of children.

Four specific characteristics of spatial knowledge should be mentioned at this stage. First is the fact that mental imagery is particularly adapted to spatial representation. Second, space is both physical and logico-mathematical in nature. Indeed, there exists a form of object-space which can be known through empirical observation. For example, one can superimpose two geometrical figures to ascertain whether they are equal in area. However, geometrical concepts finally go beyond empirical experience and fall within deductive activity. Spatial coordinates (horizontality, etc.) are a good example of the logico-mathematical or geometrical nature of spatial concepts. The coordinates are not directly perceived properties of objects. An object is horizontal or vertical only if a relationship is established between this object and other objects, including the subject's own body. Coordinates are applied by individuals to spatial reality in order to organize it, i.e., to localize objects in relation with one another. Though essentially logico-mathematical and not directly perceived, spatial coordinates are also, to a certain extent, physical in nature, like most spatial concepts, since they correspond to some properties of objects, such as the verticality of a standing body or horizontality of the water level in a tilted container.

A third characteristic of spatial knowledge lies in the fact that two distinct modalities can be used for solving spatial problems. First, spatial intuition which, although based on mental imagery, in fact goes beyond this latter in that it results in anticipation of transformations and the solving of simple problems. For example, spatial intuition can be more or less elementary, as it is in the 7-year-olds who can apprehend correctly proximity relations. There are also more evolved forms of spatial intuition, e.g., intuition of ancient geometricians who discovered theorems without being able to demonstrate their foundations. The second modality is deductive geometry, i.e., a chain reasoning allowing for demonstrations and deductive discoveries. In order to conclude this list of the specific characteristics of spatial knowledge, one further distinction should be made between two types of relationships introduced by the subjects: On the one hand an intrafigural analysis comparing the various elements of a single figure, without any external spatial reference; on the other hand, an interfigural analysis which is developmentally much later (approximately 9 to 10 years) and which results in the construction of spatial coordinates.

There were three major epistemological issues raised by this research at the Centre. The first deals with the role of mental imagery in the geometrical sphere. Imagery may be fundamental due to the fact that mental images are spatial in nature. Can it be postulated that the mental image is the principal generator of geometrical intuition? The second issue concerns the relationship between physical space and logico-mathematical space. Can it be said that one engenders the other (i.e., that logico-mathematical spatial concepts are drawn from physico-spatial concepts or vice-versa)? Or, on the contrary, should it be surmised that these two forms of spatial knowledge either develop in interaction or autonomously, merely corresponding structurally? Another important issue raises the question of the relationship between spatial operations on the one hand (used in solving geometrical problems) and, on the other hand, the logical operations which are involved in number and class reasoning, etc. Are there specific spatial operations or does geometrical reasoning merely constitute an application of logico-mathematical operations to the spatial reality? In fact, when the research on space began at the Centre, this question had already been answered; indeed, the data reported in The Child's Conception of Geometry (1960) demonstrated that measurement, for instance, does not represent a direct application of number to space.

Solutions to the problems raised above, as well as some comments that are somewhat less epistemological in nature, will be summarized at the end of this paper. But first and foremost, four of what constitute probably the most interesting experiments in the Centre space research will be discussed. Brief reference is made also to a few other experiments in the following two remarks and in the concluding remarks.

Two Remarks on the Origin of Spatial Knowledge

At the Centre symposium on space, a discussion took place on the problem of what were the basic elements upon which geometrical thought is founded. Greniewski, a Polish logician, distinguished two trends in the history of geometry. One trend was to consider points as the basic components and lines and planes as classes of points. The second tendency was to consider bodies as the basic components and areas, lines and points as abstractions of these bodies. It is the second tendency which seems to be in conformity with data drawn from developmental psychology. An experiment reported in the Child's Conception of Space (Piaget & Inhelder, 1956, chap. 5) shows that to conceive points as the smallest elements of geometrical figures requires the seriation of figures and the partitioning of lines. As of 7 years, the child begins to represent points as the smallest elements of figures, but these points are conceived of as having an area and being in finite number and related to the shape of the whole figure.

On the other hand, some results of the research work on space carried out by Piaget's Centre indicate that bodies or physical objects seem to be the basic elements of the knowledge of space, from a psychological point of view. In effect, the cognitive activity of the 4- to 5-year-old in space is characterized by a lack of differentiation between the geometrical and physical aspects of space. This was striking in research done by Vurpillot (1964) on the child's materialization of geometrical figures. She based her experiment on the well-known test of discovering simple figures (i.e., a square or a triangle in a complex figure such as shown in Figure 1). The children's drawings showed that, until 5 years,

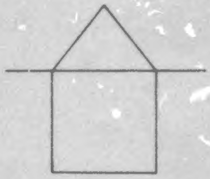

Figure 1. Simple figures contained in complex figures.

all figures are conceived of as material objects: Straight lines are totalities within which no segmentation can be made. Moreover, one line cannot belong simultaneously to two figures. In summary, each figure is treated as a material object, rather as if it were a sheet of metal, constituted by both lines and enclosed areas. The intersection of lines is beyond the child's comprehension since he interprets it as the superimposition of two material objects. For some problems, such a tendency can even be found in the 8-year-old.

TWO EXPERIMENTS BY P. GRECO ON THE OPERATIONAL NATURE OF GEOMETRICAL REPRESENTATION

The progressive achievement of a complex intellectual organization allowing for the anticipation of spatial transformations has been studied by P. Greco in two research studies that will be reported now.

Developmental Study of a System of Mental Imageries of a Spatial Group of Transformations (Greco, 1964a)

This experiment studied how representation of displacements are organized over the course of cognitive development until they finally form a logical system (the group-like Kleinian structures). The task involved mental imagery and not operational deduction. The spatial transformations that the child has to represent involve everyday activities, e.g., turning pages, placing a drawing upright, opening a lid, etc. Three transformations are involved: R, a 180° rotation; H, reflection in a horizontal axis; and, V, reflection in a vertical axis.

Method

The subjects were divided into 5 age groups with 20 children in each: 6-7 years, 7-8 years, 8-9 years, 9-10 years, and 11-13 years.

The materials were cardboard letters representing the letter p, a grey cardboard rectangle, and a sheet of paper where the subject drew his replies. The same experiment was carried out with different materials (a star, a box whose sides were different colours). But the conclusions that can be drawn from the results are analogous, and therefore we shall merely deal with the material "cardboard letters."

The experiment involved 2 types of problems:

1. Anticipation of the result of transformations. A letter is presented to the child, and both the experimenter and the child describe the letter which is then placed in front of the subject. Then the transformation (either R or H or V) is both described by gestures and by actually carrying out the transformation with the cardboard rectangle. The subject is then asked to draw (or as of 7-8 years to describe verbally) the letter as it would be after the transformation. The subject is asked to anticipate the result of V, H, R, and of the combination of H and V, each time using the letters p, b, d, q.

2. Reconstitution of transformations. Two letters, for example p and d, stuck on a piece of paper are presented to the child who has to discover which sort of transformation has been carried out in order to obtain the second letter from the first. In order to give a reply, the child is invited to "turn" or "rotate" a letter similar to the letter before the transformation has taken place. For the initial figure p, the child has to reconstitute the transformations which have been given in order b, d, q. A similar problem is used with the initial figure d, in order to result in b, q, p.

Summary of Results and Discussion

In general, the errors, which consist in substituting one transformation for another, can be classified into the following three categories:

1. The child confuses the axes around which the displacements were made (V instead of H, or H instead of V).

2. The transformation comprises only one of the inversions (left-right, V, or top-bottom, H) instead of both of them, that is, the transformation R.

3. The inverse of error 2 occurs: A unidimensional inversion is generalized to both dimensions. It should be noted moreover that sometimes the subjects leave the figure unchanged.

The proportion of successes increases regularly with age. As far as anticipations are concerned, success frequencies are the following: for the V transformations, 50% at 6-7 years; for the other transformations, 40% at 7-8 years; for single transformations, 75% as of 9-10 years; and for the double transformation, 65% as of 11-13 years.

As far as reconstitution of the transformations is concerned, 80% of success level can be observed at 6-7 years for the transformation V. As regards the other transformations, success levels are of 20 and 30% at 7-8 years and still only 40% at 9-10 years. Around 11-13 years, the frequency is over 60%.

Anticipation or reconstitution of the transformation V can be seen to be earlier than the other transformations. When a change in the material is made (box or star), the results are also different. However, it is difficult to decide which material is easier.

Greco analyzed the behaviour observed into six stages during which the representations of displacements are gradually coordinated.

Stage I (6-7 years). Type 1 errors are rather frequent. The child is aware that "it changes sides" without being able to discern whether it is from left to right or from top to bottom.

Stage II (7-8 years). On the one hand a quantitative progress is observed and on the other a new type of error appears: As far as anticipations are concerned, errors of type 2 and 3 predominate. This denotes a higher level of organization of figural intuition, but when a child analyzes figures, he tends to understand what changes and not what remains invariant.

Stage III (8-9 years). At this stage significant progress is observed simultaneously with striking regressions. The success levels with the different items are much more homogeneous, and children are able to describe the trajectory of a part of the transformed figure (i.e., the loop of the p). These two factors indicate that transformations are conceived of as changes of position instead of specific physical actions such as turning around or turning over as was the case with younger children. However, the interfigural point to point relation introduced by the child sometimes renders the problem more complicated and results in a greater number of errors for certain situations as compared to younger subjects.

Stage IV (9-13 years). Success rates for all items indicate the existence of a system in which a simultaneous representation of the three transformations is made. All the compositions between the various transformations are possible (in Stage II, only H composed with V was sometimes understood). Greco then undertakes a formalized description of the transformations, and he shows that they involve a group-like structure of four transformations (R, V, H, and the identity operation).

In conclusion, this experiment does not make it possible to trace a frontier between the role of mental imagery and the part played by mental operations during the developmental steps that lead to the comprehension of spatial transformations. It is also impossible to dissociate in all cases what is due to the material, to the transformation, and to the developmental level of the subject, though it is evident that the three factors play a role. Therefore, in some respects, the results of this experiment are not very clear. However, Greco's experiment did show that the child gradually becomes systematic or in other terms that his analysis of figures becomes more and more organized during development. This analysis can be looked upon as a gradual coordination of represented transformations. When completely developed, the child's organization of spatial representations is isomorphic to the logical operations of the formal stage, since the transformations V, H, and I (where I is the identity transformation that causes no change of position to the figure) form a group of four transformations some of whose properties are the following:

1. Involution. $V \times V = I$; $H \times H = I$; $R \times R = I$.
2. Commutativity. $V \times H = H \times V$; $V \times R = R \times V$; $H \times R = R \times H$.
3. Composition. $V \times H = R$; $V \times R = H$; $H \times R = V$; $V \times H \times R = I$.

There is no indication that the formal logical operations, which also form a group of 4 transformations, are merely applied to a spatial context (e.g., static images of various figures). Several facts tend to prove the contrary. First, the development is very gradual. Moreover, the subject's behaviour is different according to the type of figure involved. Last, the gradual coordination of transformation runs parallel to the gradual passage from intrafigural analysis to interfigural analysis.

The Gradual Organization of Spatial Representations of a Complex Figure: The Moebius Ring (Greco, 1964b)

This research is aimed at problems similar to the preceding one, i.e., studying the nature of representations (images) within a spatial context and their gradual organization. Greco endeavored to discover whether figural intuition is sufficient to organize spatial representations, or whether these latter involve coordinations analogous to those constructed by logical -mathematical operations. The subject is asked to represent the order of a series of colours on a Moebius ring (which is made by twisting a strip of paper and then joining its ends), this representation being asked to the subject after the ring is cut and laid flat in front of him. The Moebius ring transforms a ribbon which has two sides into a surface with a single side.

Method

The subjects were distributed into eight age groups between 6-7 years to 13-14 years. The experiment was divided into two parts.

Part one. The subject is presented with a Moebius ring with one single twist, the surface of which is divided into four equal areas but different in colour, i.e., A (blue), B (green), C (yellow), and D (red). The child describes the ring, manipulates it, and names the colours. Then the ring is placed in front of the child who can see colours C on his left and B on his right. The child is told that the ring will be cut along the line which divides C and B and then unfolded and laid flat on the table. This is demonstrated to the child with another ring which is not coloured. Then the child is asked to draw the order of the colours as he anticipates they will be on a strip of paper.

Part two. The procedure is the same as the first part except for two variations: (a) the questions are asked for a two-colour ring and then again for four-colour rings; (b) after each reply (the child actually makes a drawing), the ring is cut and the child can verify his answer.

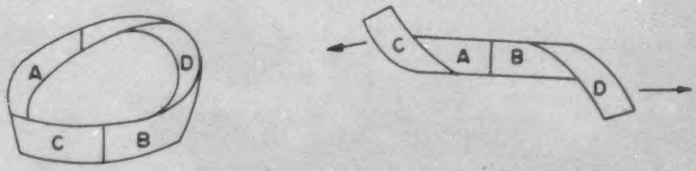

Figure 2. Coloured Moebius ring, uncut and cut.

Summary of Results and Discussion

In the description of the results, each colour will be represented by a letter. The alphabetical order corresponds to the order of the colours on the unfolded ring. The sign / makes a distinction between the front and the reverse side of the drawing. It should be recalled that the subject sees in front of him colours C and B. The correct solution is AB/CD. All solutions correct from a topological point of view are considered as successful (CD/AB, BA/DC, DC/BA).

The success rate curve shows an increase as of seven years and a temporary regression at 10-11 years. Around 9-10 years there is a 50% success rate for the first ring and 70% for the following rings (second part of the experiment). Between 11-13 years the success rate increases to 80% for the first ring and at least 90% for the following ones.

As far as progress in the course of the experiment is concerned, the presentation of the two-colour ring and the empirical verification on the cut ring is of no help up to 8 years; as of 8-9 years, however, 60% of the subjects gave better answers after the verification and the presentation of the two colour ring item.

Greco classifies the responses into 5 types of increasing complexity and correctness, some of which are very dispersed in the various age groups.

Type 0. (6-7 years) Purely figural copy. In order to decide on where to place colours, the subject looks at the ring and follows the perceptive order, without making a distinction between the apparently internal and external sides. Examples are, CABDB (since one and a same colour can occur several times in the young children's drawings), or CBDCA (which is a literal translation of the order observed visually).

Type I. (7-11 years, still 50% at 10 years) Proximity and succession. The child takes into account one of the spatial properties of the object, i.e., succession, and studies that aspect systematically on the ring. Examples are: ABCD or CBAD.

Type II. (Above all at 7-8 years of age, but not very frequent) Distinction between side-up and side-down. The colours are distributed on both sides of the strip. This means that a distinction is made between both sides of the unfolded ribbon, but still without coordination between the regions of the ring and those of the ribbon. Examples are: BC/AD or AC/DB.

Type III. (As of 7 years, but involved in many age groups) The beginning of coordination. The subject takes into consideration the place where the ring is cut and the contiguity of colours, but not the side-up/side-down relationship. An example is AB/DC.

Type IV. (Majority of subjects as of 11 years) Coordination of relations. Solutions are correct and obtained through a triple coordination of the dichotomy (colours on two different sides) resulting from the transformations of the proximity relations and the side-up/side-down relation. These correct solutions usually involve a systematic exploration of the ring, e.g., taking each region between thumb and index finger.

Greco was surprised at how difficult this experiment turned out to be, i.e., it is not solved around 9 years, although at that age many spatial notions are acquired. At first sight, the correct solutions did not seem to require very complex operations. Indeed, the reasoning involved concerns proximity and separation relations, which are recognized very early by children. Furthermore, the solution seemed to ask for a mere multiplication of relations (e.g., A is simultaneously next to B and on the back of C). Finally, the task seemed to involve figural rather then geometrical procedures.

However, the difficulty of the task is not due, according to Greco, to the unusual figural properties involved, but rather to the need to build up a system which would coordinate all the relations. Certain relations are not easily noticed (for instance that A and C are actually on the same portion of the ribbon but on different sides of it), and, above all, there are no interrelations between these and other ones which are apparently also simple (for example, that A and E are contiguous). The hierarchy of solutions proposed by the subjects bear witness to the fact that there is a gradual coordination of spatial relations whose composition must be operational in nature before a correct solution can be reached.

TWO EXPERIMENTS BY VINH BANG ON THE RELATIONS BETWEEN PERIMETER AND AREA

By proposing problems dealing with the relationship between the conservation of perimeters and the conservation of enclosed area, Bang endeavored to study the relationship between geometrical intuition (i.e., the early inferences on a geometrical content, based on mental images) and the operational deductive activity which, at some stage, goes beyond the observable spatial properties. These experiments were aimed at understanding the role of mental imagery in intuitive solutions to geometric problems. Bang endeavored to discover whether geometrical intuition depends merely on mental imagery and whether it stimulates progress in geometrical reasoning or, on the contrary, hinders such progress. Problems of representing the transformation of a geometrical figure with a constant perimeter give rise to conflicts between spatial intuition and operational deductive activity, as Bang puts it. (What he means is a conflict between empirical evidence and partially or totally logical deduction.) Furthermore, such problems shed light on the nature of the geometrical operations which are necessary for understanding the transformation of any given space. Such operations are the ones which allow the child to understand the following: (a) the continuity of the transformation, which calls for intermediate states between the initial figure (e.g., a square) and the limit of the transformation (e.g., the disappearance of the area when both longer sides of the rectangle touch each other); (b) seriation of states of the transformation according to the relations smaller than or larger than; (c) the limit of the transformation, e.g., the area of the initial square gradually becomes reduced to zero if the perimeter remains unchanged and if two parallel sides of the initial square are lengthened; and (d) the relationship between the conservation of certain elements (e.g., the perimeter) and the variation of other elements (the area) across the transformations. Bang studied these spatial (or geometrical) operations and the problem of the relationship between geometrical intuition and operational deduction in two experiments.

Representation of the Transformation of a Geometric Figure Whose Perimeter Remains Constant (Bang, 1965a)

This experiment was aimed at studying how the child represents successive transformations of a figure F into another figure F' which is obtained gradually by modifying the shape of the perimeter of F, the length of which remains unchanged. For instance, a square, which is shaped with a 40 cm long thread placed around four pins, is gradually transformed into rectangles which are carried out until the area is nil, i.e., when the longer sides of the rectangle touch one another.

The subject is asked to anticipate, to observe, and to explain on one hand the displacements of certain points of the figure across the transformations, and on the other hand the size of the enclosed area. The problem is to see how the child manages to disassociate the properties of area and perimeter. Although the two are physically linked, each has distinct geometrical properties, the perimeter being conserved during transformations whereas the area varies.

Method

The population covered children between 8 and 14 years. The interview with the child is of an exploratory nature, and the procedure varies somewhat according to the situation presented. Generally speaking, the child is shown in which direction the variation will take place by saying that "one is going to move these pins which are holding the perimeter towards this side," and the subject is asked either to draw the position of the pins (i.e., the angles), the form of the transformed figure, or both of these. The child must also state whether the area remains unchanged or not. Then he is shown an initial state of the transformation, after which the questions deal with the ensuing transformations, up to the limit state, which has area zero and back to the initial figure. Bang used eight different situations, but did not present all of them to the same subjects. The situations were the following:

Situation I. The transformation is as follows: The thread around one pin is pulled, in order to study the compensation of complementary length and the conservation of the total length.

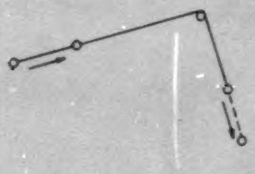

Figure 3. Situation I.

Situation II. Here again the problem does not yet deal with area. A thread forming an angle is held in place by three pins. The child must anticipate the successive positions of the pin at the summit angle if it is displaced on one side and if the thread remains tight and attached to the base AB (see Figure 4). Apart from drawing the positions (which shows whether he represents correctly the displacement according to an elliptical curve), the child must decide whether the length of the thread is conserved.

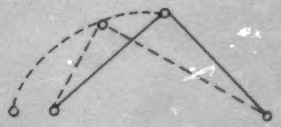

Figure 4. Situation II.

Situation III. In this situation, there is an elliptical displacement of the summit of a triangle (see Figure 5). The initial figure is the same as in Situation II, but here the triangle itself is taken into consideration and questions on the conservation of the area after transformations are asked to the child. Cardboard triangles which represent certain intermediate states of the transformation are at the subject's disposal. The situation makes it possible to see whether the child is aware of the fact that the conservation of the perimeter does not imply the conservation of areas and that, since the triangle keeps the same base line, the height is graudally decreasing until the nil limit.

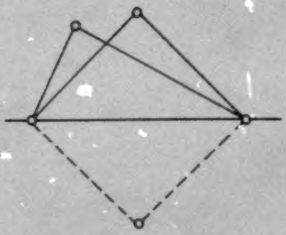

Figure 5. Situation III.

Situation IV. In this situation, a diamond is flattened. The initial figure is a diamond made out of brass rods held together by rings; it is gradually flattened (see Figure 6). Each of the sides remains unchanged, and the situation is meant to discover whether erroneous conservations of area are provoked, either because the perimeter remains unchanged, or because the child thinks that there is a compensation of the dimensions of the diagonal lines.

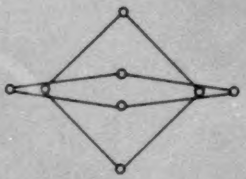

Figure 6. Situation IV.

Situation V. The square is transformed by displacement into a parallelogram. The initial figure is a brass wire square, one side of which cannot be moved; the square is transformed into a parallelogram which is gradually flattened (see Figure 7). The child can check rather easily that the area is decreasing (for example, the triangle OB'C is smaller than the part which is not covered by the parallelogram in the area of the square).

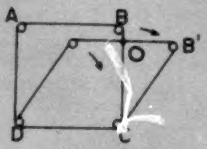

Figure 7. Situation V.

Situation VI. The square is drawn out into a rectangle (see Figure 8). The initial figure is formed of a brass chain, and the subject has available squares of four cm in order to measure the areas. Both the areas and the perimeters (number of rings in the chain) can therefore be decomposed into separate units.

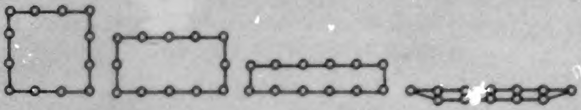

Figure 8. Situation VI.

Situation VII. A square made of a thread is drawn out into a rectangle until the limit area is nil, as in the previous situation (see Figure 9). Here the comparison of the area is carried out by cutting up the areas in cardboard.

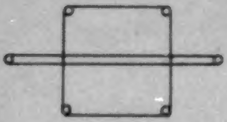

Figure 9. Situation VII.

Situation VIII. Constant areas are constructed with a varying perimeter (see Figure 10). The perimeter is constituted by a string with a sliding knot which makes it possible to vary the available length. The problem is to obtain rectangles which have the same area as the initial square. After the subject has made anticipations, he is shown a series of rectangles of equal area (100 cm^2) and decreasing height. This situation makes it possible to see whether the understanding of the relations between perimeter and areas in Situation VII allows a correct solution when the terms are reversed (constant area, varying perimeter). Furthermore, the subject should be able to discover that finally, at the limit state of the transformation, the perimeter is of infinite length.

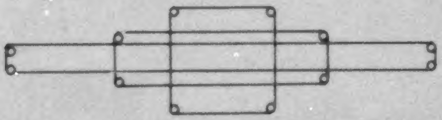

Figure 10. Situation VIII.

Summary of Results and Discussion

The behaviour patterns observed can be classified in three major stages.

Stage I A (5-6 years). Geometrical figures are conceived of as a static physical objects, and all transformations are represented as displacements.

Stage I B (7-8 years). Problem I is successfully solved. As far as the other ones are concerned, the transformation implies for the child a variation of all the dimensions of the figure (see Figure 11), in general all of them becoming smaller. The child cannot accept the limit of the transformation (i.e., the area does not disappear, but "it remains hidden under the string").

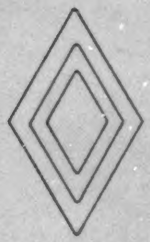

Figure 11. Stage I B response to change in one dimension--changes all dimensions.

Stage II A (8-10 years). (The notions of conservation of length and area are already acquired.) The subject has a tendency to think that the area remains constant, since the perimeter does not vary. Sometimes areas are classified into two types: (a) those which are equal to the area of the initial figure, "because the string is the same"; and (b) the areas of states very close to the limit of the transformation. The latter are conceived of as smaller than the initial area "because the string has been drawn (or flattened) too much."

Moreover only a partial series of the states of the transformation is constituted, i.e., a few states at the beginning or at the end of the transformation are linked together. As regards Situation II, the subjects anticipate the displacements of the summit parallel to the base, and then they discover that the thread has become too long and therefore they decrease one of the sides (see Figure 12).

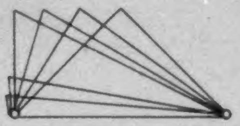

Figure 12. Progressive variations of length of one side to discover maximum area.

Stage II B (10-11 years). In the drawings, the child respects the invariance of the length of the sides of the figures which are being transformed (Situation IV and V) as well as the compensation of the dimensions of the figure, i.e., the height and the length. In general, areas are conceived of as invariant because on the one hand the length of the perimeter has not been modified, and on the other hand the child deems that the dimensions compensate each other. In fact, the compensation is very qualitative in nature and involves an addition and a subtraction: "If we take away what is in excess and add it there where it is missing, we will have the same area." None of the aspects aimed at facilitating the solutions (change in height only, for the triangle, or empirical verification of the size of the area, or the use of units) helps the child in finding a correct solution. When the child observes empirically the disappearance of the area at the limit of the transformation, this usually gives him the idea of a progressive decrease of areas, but only as concerns states near the nil state. In other words, this is merely a partial seriation, the remaining transformations still do not affect the area.

Stage III (as of 11-12 years). Subjects are able to completely seriate the transformed figures, because they can represent the continuity of the transformation and anticipate the nil limit. As far as Situation VIII is concerned (when the area remains constant while the perimeter increases), children are still encountering difficulties in representing the limit of the transformation, i.e., the perimeter can be infinitely long.

In conclusion, the development of behavioral patterns with age reflects the different levels of logical deduction. First, the role of the notion of conservation can be witnessed. The 5-6 year old, lacking this notion, is not able to imagine the transformation, that is, the modification of the initial figure with a possibility of returning to that figure. At 6-7 years, the variation of one parameter leads to the idea of nonconservation, except in the case of opposite variations (Situation I). As of 8 years, since the notion of conservation has just been acquired, this is generalized and implies that when the perimeter remains constant, the area will be too. It is only towards 11-12 years that the compensation through multiplication of dimensions enables the child to quantify areas.

As far as the nil limit of the transformation is concerned, representation of this is only possible as of 8 years, whereas at 10 years, the children can conceive of it as the final state of a limited series of a continuous transformation. Towards 12 years the limit figure becomes a logical necessity enabling the child to deduce that perimeter P and area are disassociated parameters; if with a perimeter P one can obtain a limit figure with a nil area, then a constant perimeter does not imply a constant area. The limited role of empirical evidence is striking in this experiment. Indeed, such evidence is insufficient to suggest the idea of a seriation of the transformational states. The 9-10 year old still feels that there are two categories of transformed figures, those with an unchanged area and those with a decreasing area. The notion that a seriation is involved runs together with the notion of the continuity of the transformation, this latter being deduced from rational thought and not abstracted from empirical observation.

Construction of the Largest Area Possible With a Constant Perimeter (Bang, 1965b)

A geometric figure of n sides (n = 3, 4, etc.) with a constant perimeter was to be constructed which enclosed the largest area possible. First, the child must find what are the conditions which enable him to obtain the largest area for one type of figure. (These conditions are: convex character of the figure, equality of sides, and equality of angles.) Then the subject must compare the maximal areas of different types of figures and this comparison should lead to the inference that the greater the number of sides, the greater the resulting area. The maximal area is thus that of a regular polygon of which the number of sides tends towards infinity. Consequently, the area of a circle is the limit area. A prerequisite to such reasoning is to understand that conservation of a perimeter does not imply conservation of the enclosed area.

Method

The population is composed of subjects between 7 and 14 years. The exploratory interview method has been used with most of them, with the exception of a small number of subjects interviewed with a standardized method.

The material is composed of spaghetti 20 cm long. This constitutes the perimeter which, broken into pieces, is used to construct the shapes. A rigid material is used in order to provoke awareness of the variability of the angles and the relative length of the sides. The subjects were also given paper on which to draw the shapes constructed with the spaghetti. The shapes can then be cut out and their areas compared.

The subjects were given the following problems, in order:

1. Construct an area with a minimum number of sides.

2. Construct a triangle with maximum area (having verified that the area can vary). The comparison is made by cutting out the figure deemed the smallest and covering the area of the figure deemed the largest with the pieces obtained.

3. Construct a four-sided figure (the length of the perimeter remaining constant) and find the conditions for obtaining maximal and minimal areas.

4. Compare the maximum of three-sided and four-sided figures.

5. Same problem as 3 but with a six-sided figure.

6. Compare the sizes of maximum areas of figures with various number of sides and anticipate the maximum area of five- and seven-sided figures.

7. Discover which geometric figure with the given perimeter results in the largest area possible.

Summary of Results and Discussion

Generally speaking this experiment sheds light on the contradictions which arise between intuitive constructions (seeking Gestalts) and deductive reasoning or at least actions guided by hypotheses (e.g., the relationship between conservation of perimeter and conservation of area or between length of sides and size of areas). Up to 9-10 years, the child most of the time relies on his intuition whereas around 10 years and often later a conflict develops between the intuition based on mental imagery and deductive reasoning. At around 13-14 years, subjects no longer rely on intuitive images, because of the coherence of their reasoning as they systematically relate the various parameters. But even at this age, the child's first solutions are intuitive in nature when he endeavors to construct and discover maximum areas. However, after comparing various maximum areas, the child is able to deduce that with a constant perimeter the greater the number of sides of the polygons, the greater the area.

Maximizing area with a constant number of sides. Almost all the subjects first think that the area of figures of the same type (e.g., squares) and same perimeter is conserved. Discovery of its variation develops with age. At 7-8 years, variation of areas emerges due to the lack of the notion of conservation. Each new construction, even when the sides of the triangle are merely permuted, implies for the child a variation in the area "because we have cut the spaghetti differently" or "because there are more pieces." As of 8 years, conservation of the perimeter implies conservation of the area. Between 8 and 10 years the child does not vary the length of the sides and almost always constructs an equilateral triangle. Furnished with empirical evidence, he does admit that there may be exceptions to his rule: constant perimeter implies constant area. Between 10 and 12 years subjects discover alone but empirically that triangles with the same perimeter can have different areas. When asked to construct a triangle with the largest area possible, the child's underlying hypothesis is a larger area implies longer sides. Thus, he breaks the spaghetti into one or two long segments, which in fact results in a smaller area. At this point, the child is very perplexed and does not know how to correct his errors. Without such a hypothesis and using an empirical method, such subjects sometimes are successful in discovering maximum area. It should be stated that children's spontaneous tendency is to break the perimeter into equal segments.

Around 13-14 years, the maximum surface is discovered by systematic variation of the length of the sides. The initial reactions are similar to those of the 8-12 year old group: erroneous conservation of areas and the idea that the maximum area will be obtained by using a long side. However, the contradictory results lead the subject to use a systematic method; he progressively varies the length of one side.

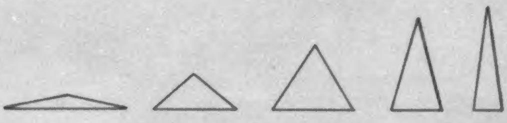

Figure 13. Progressive variation of length of one side to "increase" area.

The first condition for obtaining a maximum area, i.e., equal length of sides is, as we have already pointed out, sometimes discovered intuitively, but as of 10 years some subjects can formulate this discovery conceptually. It is only around 12-13 years that children discover the second condition, i.e., that the angles must be equal, or in the child's terms that the segments must be placed "regularly."

Maximizing area with the number of sides varying. At 7-8 years the child relies on perceptive comparisons between two figures. A figure is deemed larger because the subject "sees" it so. The result of this kind of comparison is often wrong. As of 9 years, the subject's comparisons cover all the series of the polygons, but the child is unable to deduce the law that the area increases with the number of sides. There are sometimes contradictions between perceptual estimations and inferential judgments such as the triangle is larger than the square because its sides are longer. Perceptually, the square if often deemed to have a larger area. The relation between area and number of sides is discovered as of 12-13 years, either by experimenting on one figure or by drawing conclusions from the differences observed between figures. An example of the first approach is as follows: The child constructs an equilateral triangle made up of 6 segments (two per side), and he enlarges the area by rearranging the segments in order to obtain a hexagon and discovers that the new figure has a larger area. He then transfers this experiment to a square which he transforms into an octogon and thus by induction arrives at the law, "if each piece is broken into two, then the new area will be larger." An example of the second approach is the following: Subjects compare the different figures constructed (i.e., triangle, square, etc.) and note that the area increases. They hypothesize the case of other figures (for instance that a hexagon's area lies between that of the square and octogon previously compared). This hypothesis verified, they then understand that "the greater the number of pieces (of spaghetti, i.e., of sides), the greater the area obtained." No subject can generalize the law and extend the deduction so as to anticipate that a circle would be the figure with the maximum area.

In summary, it can be seen that an inituitive approach does lead to correct solutions around 9-10 years (constructing maximum area) but that a more reasoned approach leads, at this age, to less successful replies. The conflict between intuition and reasoning is only surmounted late in development, since well developed deductive capacities are necessary to discover the geometric properties. Bang concludes that the latter capacities are not discovered directly through mental imagery.

GENERAL DISCUSSION

The results of the different research on space conducted at Piaget's Centre for Genetic Epistemology will be discussed first from an epistemological point of view then with regard to some of their psychological and educational implications.

Epistemological Implications

Mental Imagery and Operational Development

Mental imagery, which is not well adapted to represent either temporal aspects of reality or logical concepts, is particularly fit for representing the spatial aspects of reality and thus plays a definite role in spatial knowledge. The similarity of nature between signifiers and significates, in the spatial field, makes the development of geometric intuition possible. This latter enables subjects to solve problems before they are able to apply a systematic method. In Bang's research on maximum area, for instance, success based on intuition is observed up to 9 years. The subject constructs a maximum area thanks to his search of Gestalt, without being able to furnish the rules behind his construction. Toward 10 years of age, subjects either succeed through an intuitive method, or fail when they use inferences not based on mental imagery (the inference being "in order to obtain a large area, you need a large side").

Proof that geometric intuition is largely based on imagery is given by an experiment of Hatwell (1964) who compares the results of blind children with those of seeing children, in a task involving the representation of spatial order after displacements.[1] Blind children are generally between 4 and 6 years in arrears of normal children, and this can be attributed to their deficit in the field of imagery (or, more generally, a deficit in figurative functions). In logico-mathematical tests as well as in verbal reasoning, these blind children are only some one to three years behind seeing children. The latter, even when using tactile perception, with a screening procedure, nonetheless remain in advance of blind children. In fact, their results are the same as when they can see the material. The understanding of changes in spatial order thus relies heavily on the capacity to represent reality spatially. Although Piaget has not stressed this point, it seems evident that even the operational solution to spatial problems--such as the Moebius ring--requires, even for adults who are not geometricians, the assistance of a representation through mental imagery.

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[1] For example, children were asked to reconstitute a "train" made of three pieces of wood of different shapes, at different spots of a non-regular circuit. At some spots, the order of the "carriages" (from a right-left point of view) is the inverse of the initial order. The material was conceived so as to make possible replies based solely on tactile perception.

In spite of the preceeding remarks, it must be admitted that mental imagery is not alone sufficient for solving spatial problems. Work on the development of mental imagery (Piaget & Inhelder, 1970) showed that the young child's images only represent static configurations. Representation of transformations is only possible when the child's operational development interacts with the image (clear progress as of around 7 years). Besides, geometrical intuition cannot be conceived of as a mere utilization of mental images, since it implies an important intellectual activity of the subjects, e.g., establishing relationships. Briefly speaking, this intuition is operative in nature and mental imagery cannot give a good "translation" of this operative character.

The Centre's research particularly established a relationship between notions of conservation, which are operational in nature, and the representation of the variations of area. Once the conservation of length and area is acquired, for instance, children make deductions which transform their representation of spatial phenomena. They overgeneralize the concept of conservation which has just been acquired and are often incapable of even taking note that, in fact, the area diminishes with the change in shape of a figure with constant perimeter. The four main experiments which have been summarized clearly demonstrate the need for operational activity in order to solve spatial problems which involve neither arithmetic calculation nor deduction. Thus, it is not until around 9-10 years that the child can anticipate correctly the result of rotations and not until around 11 years that he can reconstitute the transformations and understand the result of multiplying two transformations. The subject needs, in effect, to construct a method which will allow him to coordinate the transformations (e.g., Greco's experiment on the letters p, b, d, q). The same applies to the Moebuis ring experiment when, although the various spatial relations (proximity, side up, side down, etc.) are easily representable, the transformation can only be represented through a very complex mental activity connecting these various relations.

Physical and Logico-Mathematical Space

To touch on the problem of the relationship between physical and logico-mathematical space, account must first be taken of an experiment by Vurpillot (1964), who shows that geometrical relationships are conceived of by the young child as relations between physical objects. From these results, Piaget concludes that the two types of space are not differentiated at this level. It could not be postulated that geometrical space develops out of physical space. The insufficiency of empirical evidence is particularly obvious in Bang's experiment on the variation of areas with constant perimeter. Geometrical concepts are acquired through the enrichment of the relationships the child establishes (intrafigural as well as interfigural relationships) and by a coordination of representations. In L'Epistemologié de L'Espace (Bang, Greco, Grize, Hatwell, Piaget, Seagrim, & Vurpillot, 1964), Piaget deems that there is no interaction between physical space (i.e., concepts and images drawn from the experience of objects) and logico-mathematical space (the spatial operations and the geometric concepts they underlie). He merely feels that there is a structural correspondence between these two types of spatial knowledge. More recently, Piaget (1974) has stressed the role of spatial operations or preoperations in physical space: Spatial properties of objects are known through spatial operations applied to objects. An interaction between the two forms of space is therefore not excluded, but it is not clearly defined. In my opinion, it would be heuristic to look upon these relations as an interaction sometimes resulting in a reciprocal assistance, sometimes leading to conflicts and even blockages, as is the case for the relations between causal explanations of physical phenomena and logico-mathematical deduction.

Geometrical and Logico-Mathematical Operations

If the relationships between geometrical and logico-mathematical operations are now envisaged, the first striking fact is that they are isomorphic. Piaget had already shown how measurement of length could be analyzed in terms of partition and inclusion operations which are isomorphic to the grouping of operations involved in number. Greco demonstrates that at the formal operational level, geometric reasoning (which makes possible understanding of rotations, etc.) can be formulated in terms of operations which are isomorphic to the "group" of operations involved in propositional logic.

However, spatial operations cannot be conceived of as merely an application of logical operations to space, since they are acquired progressively without direct synchronism with the development of logical operations. Furthermore, geometrical reasoning seems closely linked to the specific spatial context.

Psychological and Educational Implications

Most of the questions discussed from an epistemological point of view can now be extended further with a special regard to their psychological and possibly educational implications.

As regards the necessity of improving representations of spatial transformations through operational mental activity, a more detailed example is brought to mind here. In order to anticipate, or even merely observe adequately, the progressive transformations of a geometric figure (Bang's Experiment 1), the child must understand the continuity of the process, and this understanding stems from the idea that some parameters remain invariant whereas others vary. The child must also seriate the states of the transformation according to the relation "smaller than" (or "larger than"). Furthermore, he must have some idea of the limit of the transformation and relate this limit state to the other states of the progressive change. Not only in this experiment, but also in all the other studies carried out at the Centre, the development of spatial representation is explained in terms of improving the understanding of transformations, such an understanding being obtained through a systematic organization of representational imagery.

As far as the distinction between physical and logico-mathematical space is concerned, it must first be noted that it does not correspond to the distinction between figurative and operative aspects of knowledge. It is true that perception, and more generally empirical observation, as well as static mental images (the figurative aspects) are mainly the product of the experience that the child acquires of the concrete spatial environment in which he lives. However, such a spatial environment is in turn understood thanks to the child's activity, that is, thanks to the operative aspects of knowledge such as displacements of one's own body or of objects, and thanks to mental activity allowing for the abstracting of certain relations or the establishing of a relationship between various points of view. Physical space therefore has an operative as well as a figurative aspect. Logico-mathematical space is essentially operative in nature, but it deals with relations specific to space, which, in their elementary forms, can easily be represented through mental images and are therefore based on figurative functions.

This distinction between physical and logico-mathematical space is of great interest concerning the learning of spatial notions. It suggests that two types of experience, or activity, should be practiced by children. First, children should engage in empirical activities where they manipulate objects in order to become familiar with their shapes, observe transformations, and make actual comparisons (like superimposing figures). The second type of activity, logico-mathematical in nature, should result in going beyond what is empirically observable. Such an activity would consist in building up a systematic method in practicing deductions and in dealing with notions that do not correspond to empirical entities. The two types of activities are coordinated at a final stage, as can be seen in the older subjects of the Moebius ring experiment: They coordinate all the spatial relations involved and have a systematic method (logico-mathematical aspect), but they also investigate thoroughly the ring (spatial properties are also discovered as physical properties would be).

At an earlier stage, the two ways of solving spatial problems, i.e., by empirical observation and action or through inferences of more or less logical nature, can give rise to conflicts. This is exemplified in Bang's experiment on maximum area.

With regard to geometric intuition, it must be stated that this concept is rather ambiguous since, even among the authors mentioned in this paper, it can convey different meanings. For some it is an insight based on Gestalt, while for others it is a nonoperational or an operational way of solving problems. The latter implies important inferential activity but no precise measurement nor theorems. This concept must therefore be used with caution. Geometric intuition will be defined here as a mental activity concerning space essentially based on figurative aspects, but involving inferences (see introduction to this paper). This geometrical intuition is necessarily involved before 11 years, since neither rigorous abstract deduction nor a very systematic method can be used by the child. Even after this age, it could be useful to rely on intuitions to introduce new spatial notions to the child.

To conclude this paper, a summary of the procedures used in the above experiments which have a training effect will be made. Generally speaking, learning spatial concepts seems strongly related to the child's attempts at representing spatial transformations.

The successive steps of certain of the Centre's experiments dealing with these transformations must undoubtedly have a training effect. These steps are the following: The child is asked to anticipate the transformation, then to observe them empirically, and then to make anticipations and observations about simpler problems before getting back to a more complex situation. A few remarks can be made about these different steps. First, asking for anticipations triggers off the mental activity of the child, whereas simple observation can induce a rather passive attitude. Furthermore, it may lead to a confrontation of the anticipations with the empirical observations. About the latter, it must be noted that their effect on the child's understanding of spatial concepts is limited. Most of the time this effect is nil, for the problems involved, before the age of 8 (e.g., Moebius ring). Even after this age when the observation becomes accurate, it is not sufficient to solve problems. However, this fact must be taken advantage of, namely that spatial problems can, contrary to temporal ones, for example, give rise to observations and confrontations between different situations.

The changes in the material used (e.g., a simplification of the situation by asking questions about a two-colour ring in Greco's experiments) is of training value mainly because it elicits comparisons between different situations. The very choice of the material of each situation presented is also important. Thus, Bang's use of a rigid material to construct the perimeters in certain situations, as well as the use of the thread suggesting continuous transformation, sometimes seems to help the children become aware of the rules of constructions.

As far as the hierarchy of difficulty of the problems presented is concerned, two points should be stressed. First, it seems favorable to make the child reflect upon simpler elements, like the compensation of the length of sides or the trajectory of one angle during the change in shape, before he is asked to represent all the aspects of a complex transformation. But this procedure should not suggest that a kind of programmed learning, evenly developing step by step, can be elicited. The fact that the child reaches a new stage of development in the understanding of spatial relations implies a new structuration of the problems and results from the coming back to previous problems and statements and from the conflicts that this comparison may provoke. A first form of conflict can occur between the anticipation of the child and his subsequent observation. The anticipations may be drawn from intuitive figurative representation or from mainly inferential processes. A second form of conflict, the most important according to Piaget's theory of equilibration, occurs between different ways of tackling the same problem. For example, one of the most effective training procedures used by Inhelder, Sinclair, and Bovet (1974, chap. 6) consisted in eliciting two different types of judging length, i.e., a judgment based on the spatial correspondence scheme and a judgment based on counting. The two types of judgment were then confronted, and they often led to a new solution of the problem. The solution was at first incomplete and consisted in a compromise between the two types of judgment. However, the solution was eventually correct and resulted from a new type of intellectual activity which can be described as a synthesis of the two previous ways of solving the problem.

The training procedures described above are consistent with the Genevan conception of learning, which entirely differs from the common views on education. According to these views, education aims at "printing" certain cognitive structures or bodies of concepts in the child's mind. There seems to be little necessity to know how this mind works and develops when it is not trained, because intellectual growth is deemed to be elicited by training experiences.

Actually, even in order to create new habits, like following the right itinerary in a maze or counting up to 100, the cognitive difficulties that the subject must surmount and the behavior patterns that he can assimilate must be taken into consideration. When the goal of learning is to help the child create new structures of knowledge which will be applied to a more and more extended field of reality and which might in turn generate new cognitive structures, it is absolutely necessary to subordinate learning to the laws of intellectual development. For example, the training of a concept like horizontality or duration can be efficient only at a certain stage of development, and it could not be done in any lapse of time. Learning procedures must take account of psychological processes that have their own speed. In stimulating certain processes observed in cognitive development, such as conflicts between schematas, Genevan training experiments have yielded good results. One interesting fact must be noticed: The subjects who find their own "wrong" solution to the problems in the course of these experiments give the best answers at the end.

The reason learning must be subordinated to the laws of development is that operational structures do not derive from structures that might exist outside the child but stem from the coordination of internalized actions. As far as logical necessity is concerned, it cannot be demonstrated empirically. For example, a preoperational child who fails to conserve number or to perform a class inclusion task cannot be "shown" that he is wrong. The same holds true for complex spatial notions, as is evidenced in the research reported in this paper. It is necessary to study the processes of mental coordination which lead to logico-mathematical and geometrical operations before undertaking training experiments. In brief, training, according to the Genevan conception, consists in trying to accelerate cognitive development.

These remarks on learning touch upon a problem which the Centre's research on space was not aimed at studying. However, the particularity of the work done by Piaget and his collaborators is to be of general interest. It allows for epistemological as well as psychological conclusions and for developments in different areas of psychology. This is what the present paper was aimed at suggesting.

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Needed Research on Space in the Context of the Geneva Group

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Research on space concepts needed to complement that which has been completed in Geneva should involve those aspects that have been overlooked as a result of the inherent limitation of the Piagetian approach. The Piagetian approach has permitted an immense step forward in the science of psychology. We do not want to imply that Piaget should have considered all the points and problems which remain to be studied. His method was, and is, the best suited to the study of basic epistemological problems as far as space is concerned. We do not question the value of the prior work by Piaget and his collaborators. Rather, if something new can be added to this work, it will require a change of method. The method suggested is not entirely different than the Piagetian method, but is an adaptation to the problems that remain to be studied. It needs to be stressed also that the structural approach of Piaget is inseparable from a developmental approach that is to be constructivistic and that implies equilibration processes.

The Piagetian explanation of spatial notions in children is mainly based on the concept of adaptive equilibration structures. A structural explanation has the four main advantages that will be listed below before discussing certain implicit limitations.

1. The genetic structural approach does not deal with the superficial level of behavior (i.e., the level of "performance"), but sheds light on the underlying mental activity. For example, instead of giving a mere description, such as "some children can construct a tower the same height of a model tower because they know how to measure heights," the structural analysis endeavors to give account of procedures for knowing, i.e., the operations of partition, seriation, iteration, etc.

2. A structural explanation thus gives an account of the operational nature of knowledge. A concept is not described as a more or less static idea, but in terms of mental operations which consist in transforming what the subject observes (e.g., imagining the reverse of the action observed, or finding different ways which would lead to the same result). These transformational capacities enable the child to apprehend the transformations of the outer world. It is only when children can begin to understand physical or spatial transformations--by knowing what changes and what remains invariant--that one can be sure that they have reached an operational level.

3. The interrelation between the different inferences of a child dealing with a certain type of problem appear clearly. For instance, the different arguments a child gives in order to justify a conservation judgment correspond to different operations all within the same cognitive structure. The explanation of spatial abilities in terms of systems of transformations sheds light, for example, on the close relation between different achievements of seven-year-olds since these achievements all derive from the capacity to establish topological relationships.

4. The structural method points out the resemblance between different fields of knowledge that seem heterogeneous when not studied with this perspective. The similarity of spatial abilities, such as length measurement, and logico-mathematical abilities, such as conservation of small numbers, appears through an analysis of these behaviors in terms of the groupings of operations. On the other hand, a precise description of the differences between notions can be given, e.g., of what is not alike in space measurement and number, or between topological and projective spatial notions.

The genetic structural approach also implies a search for what is common to children of the same developmental level and what is common to different ideas or judgments. Because of this particularity of intent and the generality of the structures used to give account of children's cognitive processes, the approach leaves some gaps in our knowledge of the psychology of space. The main limitations of the Piagetian approach to the study of space appear to be the following:

1. The role of the object of knowledge or what the subject assimilates as external reality does not appear clearly from previous studies. Though Piaget's theory stresses the fact that knowledge stems from the interaction between the subject and the object, the role of the object seems most unclarified (see Smock & von Glasersfeld, 1974; von Glasersfeld 1975). A careful analysis should be made of this interaction. In such an analysis, the parameters of a situation must not be studied per se, but with reference to the conceptual framework of the subject facing the situation.

2. Piaget's approach accounts for a very limited number of stages of cognitive development. It is necessary to describe many more substages in that development in order to clearly understand the process and to gain better knowledge of the different notions constructed during childhood and adolescence.

3. The mental operations forming groupings or group structures give a model of the abilities of children, i.e., all the possibilities of reasoning they have, but not the actual process (strategies) of solving problems. A study of the process itself would clarify the interrelation between the child's cognitive structures and the attributes of a situation; also, it would give invaluable information for educational purposes.

4. A metaphor can best specify the results of the cognitive studies conducted in Geneva. A forest of scattered trees, each tree representing a particular concept, has been found. We now need, beside studying the growth of more concepts, an understanding of the connections between these concepts. These types of studies are particularly important since there is strong evidence that the relations between different notions (or fields of knowledge) are major factors in promoting intellectual development (Inhelder, Sinclair, & Bovet, 1974; Piaget, 1974).

To make the prior body of research on space more complete the next phase should use some of the new methods developed in Geneva and should be applied to the study of this domain. In addition, types of experiments that have not been conducted in Geneva are necessary. The aim and the methodological principles of three main types of research will be suggested.

Interaction of Cognitive Structure and Context Variations

The aim of this type of research would be to determine how the child assimilates different aspects of space, i.e., which spatial parameters influence his spatial judgments. The procedures would include both techniques for facilitating correct answers and, on the contrary, eliciting errors by providing materials which may stimulate the development of spatial notions. One example relevant to these problems (i.e., varying the parameters of the situation) is the experiment on the concepts of duration by Montangero (in press). A summary of this experiment is presented in order to exemplify the method suggested.

In his book on time Piaget (1969) demonstrated that operational judgments of duration are based on the speed of observed changes and the amount of "work" done (distance covered, number of objects handled, etc.) during the duration which is evaluated. Further, it was hypothesized that, in addition to considerations of speed and work, the relationship established between the relative temporal order of starts and stops[1] played an important role in operational duration judgments and in their elaboration. One of the experiments designed to test this hypothesis as well as to define substages of the development of the concept of duration, consisted in varying the parameters speed, work done, and temporal order. The design is summarized in Table 1.

Table 1

Summary of Design for Studying Variation in Context and Judgments of Durations

Speed and work done / Temporal Order	No cinematic aspect: turning lamps on and off	Speed consists in frequencies: discrete actions: putting beads into a container		Speed consists in displacements: continuous movements of dolls or toy cars	
		Perceptible results: glass container	Nonperceptible results: opaque container	Distance covered comparable cardboard tracks	No permanent trace of distance covered
Synchronism: events start and stop simultaneously	1	4	7	10	13
One event starts before the other; simultaneous stops	2	5	8	11	14
Same duration, no simultaneity: same time interval between relative starts and stops	3	6	9	12	15

Note: The numbers in the table refer to 15 different experimental situations.

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[1] For example, the awareness that two events began simultaneously but one of them ended before the other.

Each subject was presented with 15 different situations (Table 1) and was asked to give a relative duration judgment[2] and to justify it. The variations of the duration judgments and their justifications were then related to the parameters of the situations in order to see what kind of relation the subject introduces (or constructs) between the different parameters. As already stated, the goal of such an analysis is not to study the importance of the parameters per se. Indeed, the results of this experiment indicated that the same situation is often evaluated very differently according to the stage of cognitive development of the subject. The presence of a certain parameter (e.g., a difference of speed without difference of work done) may have no influence on younger children's judgments but may become an important cue for the older subjects when they evaluate duration. Therefore, comparing the subjects answers for different situations does not exclude developmental analysis or comparison of answers at different levels of development.

The analysis in term of relations established by the subject permitted a better understanding of the child's reasoning about duration. Different modes were used to solve the temporal problems, and the possibilities and limitations of the children's judgments at different substages of conceptual development were assembled at the same time. Similar methods applied to spatial judgments should yield the same kind of information about space concepts. The varying parameters could be different geometrical relations or different types of presentation allowing for different ways of apprehending similar relations. Such a method would be particularly suited to the study of the interrelation between "physical space" (spatial properties of objects discovered through active observation) and "logico-mathematical space" (spatial properties constructed by the subject and discovered through deduction).

The method of varying the context sheds light on what underlies each child's particular judgments. Comparisons can be made between contexts with other judgments and with behaviors observed at different levels of development. These comparisons facilitate the discovery of the status and causes of apparent regressions in intellectual growth. For example, in the study of time, three types of regression were observed, which in every case obscured actual progress in cognitive capabilities.

1. The early correct answers do not indicate the presence of a real understanding of the problem; they stem from a simplifying assimilation of the problem. For example, five-year-olds tend to judge duration from only one aspect of the situations, namely the final temporal order. This simplification sometimes leads to correct answers

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[2] The subject was asked "Did the two events take the same time or did one of them take a longer time?"

which, actually, are not duration judgments and therefore should not be considered as correct answers.

2. At a certain level of development, children may encounter more difficulties in solving a problem than they had previously because of the progress in an element (concept) of the "field" of ideas related to the specific concept under study. For example, as the concept of length develops, a temporal situation becomes more complex for the subject, i.e., whereas, before he only took into consideration the places of starts and stops.

3. Apparent regression may also be due to the fact that children vacillate between two different modes of evaluation to solve a problem. In the duration studies, children often would alternate between a mode of evaluation based on the comparison of temporal orders and one based on the spatio-cinematic content of events. These altering centrations explain the fact that certain situations correctly evaluated by a child using, for example, comparison of temporal orders, are incorrectly evaluated a few months later because of exclusive centration on other parameters. All such regressions diappear as the evaluations achieve the operational level.

The question of regression is discussed here because in several Piagetian spatial tasks a depression of performance is observed around the age of nine. Yet cognitive progress appears to be continuing. Thus, these regressions should be investigated and the method of context variation appears the best method for teasing out the relevant variables.

Interaction Between Spatial and Logico-mathematical Operations

Geometrical abilities can undoubtedly help to understand mathematical problems. On the level of scientific thought, it seems that mathematicians often have recourse to spatial intuition (Beth & Piaget, 1961). As far as the child's thought is concerned, the research completed in Geneva shows that spatial and mathematical notions are undifferentiated in young children (which is one reason for the failure to conserve small numbers). Later in development, the two types of notions are dissociated, but their structural aspects are isomorphic (Alonzo, 1970; Bruce, 1968; Leskow & Smock, 1970). There is evidence of ability to solve certain spatial problems without a rigorous deductive method, with the aid of imaged representation (i.e., spatial intuition) in the child when mathematical intuition is very limited. Consequently, it is not unexpected that many teaching methods of mathematics rely heavily on a spatial representation of mathematical problems.

On the other hand, geometrical abilities, though they are based on specific spatial operations, also depend on general logico-mathematical operations such as seriating, represention of all the possible combinations, etc. The possibility of using a *systematic method* is closely related to the logico-mathematical achievements of the formal operational stage (Inhelder & Piaget, 1958). When children reach this developmental level, they are able to apply such systematic methods to spatial problems (Alonzo, 1970; Bang, 1965; Greco, 1964; Leskow & Smock, 1970).

The study of the relations between logico-mathematical and spatial concepts should not overlook either of the following points:

(a) influence of progress in geometry on logical or mathematical notions, and

(b) influence of progress of logico-mathematical notions on the development of spatial notions.

The bi-directional influence could be studied by methods comparable to the learning studies reported by Inhelder, Sinclair, and Bovet (1974) concerned with the reciprocal influence of mass conservation and class inclusion. Progress in inclusion (comparison of the subclass and the total class) obtained through induced improvements in conservation of mass, but the reverse did not hold. It would be very interesting to test, by a similar method, the reciprocal influence of mathematical and spatial acquisitions which normally take place approximately at the same time in cognitive development change.

The first step of such research would be to find suitable training procedures, i.e., exercises which are efficient in accelerating the acquisition of a concept because they take account of the laws of development.[3] The elaboration of a suitable training procedure for a concept goes through the following three steps:

1. Cross sectional developmental studies are needed to define the different components (i.e., the set of inferences and operations) and stages of acquisition of the concept.

2. Different strategies that children use to try to solve a problem involving the concept need to be identified. For conservation tasks, Inhelder, Sinclair, and Bovet (1974) presented the problem sometimes as discontinuous, and sometimes as continuous. The authors could distinguish in the young child two ways of evaluating the relative length of segments:

(a) by comparing the extremities of the segments (ordinal evaluation based on the "frontier" effect) or

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[3] Development is considered (according to Piaget) as a gradual coordination of schematas achieved due to the child's organizing activity and, is always regarded as an equilibration process.

(b) by counting the number of discontinuous elements forming the segments.

3. Exercises need to be devised comprising actions, or preoperational inferences, or operations that the subjects can perform. This implies that the optimal level of development of the subjects has first been determined and the operative capabilities of the subjects are known.

As the main goal of a training procedure is to stimulate the subjects mental activity vis-à-vis a particular problem, the best procedure consists in presenting different situations or types of exercises under conditions that permit comparison among those situations. In the learning study of length mentioned above, subjects had to perform the same task (constructing a segment with matches whose length had to be equal to the length of a model segment) in three different situations. The first tended to elicit an ordinal comparison of length (evaluation type (a) above), while the second situation tended to promote a numerical comparison (evaluation type (b)), and the third allowed for the discovery of the difference of units used in the model segment and in the child's construction. The experimenter asked the subject to return to the first situation immediately after he finished the second task, and to return to the second and first situation after the third task was completed (the three situations always remained in the child's perceptual field). Such a procedure which permits comparisons between different ways of evaluating a parameter, often elicits conflicts between these modes and consequently important improvements in the comprehension of the problem.[4]

Space and time, which are reunited in a single concept in relativist physics, are interrelated during the course of intellectual development. Piaget's book on time (1969) demonstrated that the operational notion of duration involves a consideration of the distance covered as related to the speed. On the other hand, the notion of length seems to be first grasped due to a representation of displacements. Thus, Piaget defines space as the coordination of displacements. At least two kinds of problems concerning the relation of space and time should be further investigated.

The first concerns the notion of distance and its interaction with the concepts of speed and time. The research conducted up to now in Geneva showed how these concepts become progressively coordinated near the beginning of the concrete operational stage (between the age of 6 and 9). The acquisition, around eight to nine years, of an operational notion of duration allows for correct temporal comparison between different events, but 14-year-olds still encounter great difficulty in establishing correct relations between distance, time, and speed when the two latter parameters must be deduced from spatial datum in a double reference system (Crépault, 1975).[5] No research evidence is yet available, however, concerning the effect of progress in cinematic notions on the concept of distance or other spatial notions. This effect, therefore, should be investigated thoroughly.

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[4] To these main principles for elaborating a training procedure must be added other considerations, such as the necessity to evaluate as precisely as possible the subject's developmental level (pretest) and the status of the progress observed (generalization at the posttest, stability of acquisition, or consolidation at the second posttest).

A second type of experiment is needed to determine how children can understand spatial relations (relative distance, directions) in situations involving points of references, or observers, moving at different speeds. Here again, spatial relations would depend on cinematic cues. This type of problem--which could be understood only at the formal operational stage--is now under investigation at Paris VII University by a team of physicists interested in the psychological foundations of physical concepts. One situation presented consisted in imagining two parachutists falling from a plane at different speeds; one of them drops his eyeglasses that the second one manages to catch. Questions are asked about the duration, distance, speed of the fall of the eyeglasses relative to both parachutist number one and to number two.

Cinematic situations are interesting for the study of spatial relations even when no questions are asked about the temporal and cinematic parameters. Many experiments could be designed on the theme of trajectories of moving objects in a plane or within a three-dimensional space A first example of such an experiment is Inhelder and Piaget's (1958) research on the equality of the angle of incidence and the angle of reflection in a game of "billiards." In this experiment, there is a close connection between physical relationships (causality relationship of the position of the propeller with the impact of the ball thrown on the aim, which consists in a wooden block) and spatial relationships (the subject can understand the physical law only by spatially structuring the game board and by comparing the angle of incidence and the angle of reflection of the ball trajectory). Similar experiments, requiring comparison of geometric figures, could be done within a football game context. The study of trajectory should provide many suggestions of experiments where both the physical and logico-mathematical aspects of space are involved. In a recent study by M. and I. Fluckiger (1975) subjects were asked to imagine the trajectories of a tennis ball hanging by a long thread fixed to a hook in the ceiling. First, the children were required to anticipate the trajectory of the ball when the experimenter, who had removed the ball from the vertical axis where it normally hangs, would release it from a predetermined place. The question was: Where should one place a wooden pin so that the ball would knock it down? At the end of the experiment, children were asked to imagine the bowl-like figure (i.e., a portion of a sphere) formed by all the possible displacements of the hanging tennis ball.

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[5] A vibrating stick prints dots on a strip of paper moved by the experimenter. In one situation, the frequency of the stick is even, whereas the strip of paper is moved first slowly, then rapidly. The subjects are presented with the printed paper and must evaluate the relative time and speed of the displacement of the strip of paper between two dots close together compared to two dots at a wide interval (end of the strip).

Research Relevant to Applications to Education

Some people incorrectly assume that results of the research in developmental psychology conducted in Geneva allow for direct applications to education. Those who understand the Piagetian studies know very well no such direct application to education is possible. An "intermediate" body of research is necessary for purposes of application. Piaget has endeavored to find out the general processes of knowledge acquisition or how better forms of knowledge evolve from early more limited forms and what underlies our basic scientific concepts. Thus, he has had to make abstractions from the individual characteristics of his subjects. Due to these abstractions, general developmental processes (such as equilibration) were discovered and the potentialities of children at different levels of development were defined in terms of cognitive structures.

When it comes to the elaboration of "good" teaching methods, the knowledge of these processes and potentialities is of great help, but certainly does not suffice. Teaching deals with unique individuals and, therefore, it is necessary to base teaching methods on the results of research aimed at specifying individual cognitive characteristics and at describing the actual process (performance strategies) involved in the utilization of cognitive structures. Two kinds of research are most helpful to serve these purposes.

First, differential studies should be conducted, not only to show that there are differences in the acquisition of spatial concepts, but also in order to analyze these differences. One question which must be dealt with is, "Do children of the same developmental level solve a given spatial problem in different ways?" As a matter of fact, even differential studies of the usual type can give useful information for establishing teaching curricula.

The second type of research can be based on the analysis of children's organization of action in order to solve spatial problems. A team of research workers directed by Barbel Inhelder is undertaking such a study in Geneva, without focusing on spatial problems. These researchers are attempting to shed light on the processes of discovery in action and on the interplay between the processes of goal oriented activity and the child's concepts and representations.

For example, Montangero (1975) has conducted an experiment involving both physical and spatial notions. In this study the materials used consisted of objects similar in shape (cubes or balls) but of different weight or different volume. There were, for instance, five-face empty cubes and normal "closed" cubes loaded with plasticine or plasticine balls and leaden balls of the same diameter. The glass containers were half filled with water, and the child was asked to put whatever he liked into the water in order to find out which objects noticeably raised the water level and which ones have little or no visible effect on the water level. In a second part of the experiment, the water level had to be raised up to a certain point by immersing objects. This point could not be reached unless the children made some changes to the material, namely by loading empty cubes with plasticine or with a leaden ball.

The actions observed and the few comments made by the subjects were analyzed with the aim of specifying:

(a) how sequences of actions can be delimited by considering the successive immediate goals of the subject,

(b) what are the relations between the different sequences, and

(c) what can explain the changes of goals and of action organization.

On the one hand, the subject's actions were related, for each sequence, to the notions or particular representation which seemed to direct the actions. (For example, children up to 9 years of age hypothesized that the weight was the cause of the raising of water level, or the children constituted couples of objects, differing by one parameter, in order to compare their effect when immersed.) On the other hand, the analysis tried to specify which aspect of the situation (characteristics of the objects, effect produced, experimenter's intervention) were taken into consideration by the subject and how each influenced the course of his activity. In this respect, this investigation asked a question which often has been overlooked in the research on cognitive structures, i.e., what is the role of the object of knowledge?

The general implications of the results of the water level experiment, briefly summarized, are as follows. First, there is a rather large variety of action patterns among children whose conceptual level seems similar. Second, this type of experiment clearly reveals how the successive sequences of actions are guided by an interaction between conceptual representations and the different cues of the situation. Third, the discovery of a means to reach the goal, in the second part of the experiment, is often preceded by successive practice of the isolated schematas which, once they are combined, allow for success. Finally, the influence of the results of the child's actions (raising of the water level) is limited and depends on whether the anticipations are confirmed or not, and corresponds very much to what appeared in a previous study on goal oriented activity (Karmiloff & Inhelder, in press).

In the experiment discussed here, the subject's conceptual level concerning the main notions involved (dissociation between weight and volume) were preoperational. In the field of space, similar research on how children use an operational spatial concept when trying to solve problems which require the acquisition of higher concepts would be most helpful.

These suggestions for research are, of course, very general. It is hoped that many research workers, knowing well the previous developmental studies of space, will transform these suggestions into actual experiments. The results of such experiments should be very beneficial to the teaching of geometry as well as for cognitive psychology.

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Cross-Cultural Research on Concepts of Space and Geometry[1]

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Apart from studies on the conservation of length, cross-cultural research on geometric concepts can be enumerated very quickly: Geometric vocabulary and measurement and estimation skills have been investigated by Gay and Cole (1967) and Ohuche (1973) in Sierra Leone, angles by Okonji (1971) in Uganda, and coordinates also by Okonji (1971) in Uganda and by Page (1971) amongst Zulus in South Africa. I have been able to find no studies dealing with geometrical reasoning or with school achievement in geometry.

By contrast, there is considerable cross-cultural research on the perceptual abilities which presumably underlie the development and application of geometric concepts. Many of the relevant results have spun off the effort to devise culture-free tests of intelligence, which have perforce used abstract visual materials. Other relevant results come from investigations of the universality of Piagetian stages of intellectual development and from studies of pictorial depth perception and representation. From these various fields of research, certain clear results have emerged. For example, whereas native Africans of all nationalities appear to be considerably retarded in perceptual development relative to Europeans of the same age and length of schooling, even illiterate Eskimos and North American Indians do not differ markedly from Europeans of the same age.

Although no research relating perceptual development to geometry achievement has been reported from developing countries, it is obvious that perceptual retardation would cause difficulties in learning elementary geometry. Skemp (1971) reports the case of a Uganda school student who, attempting to illustrate Pythagoras' theorem, produced the diagram shown in Figure 1; this child's inability to deal with obliques would certainly make it difficult for him to comprehend this particular interpretation of the theorem. Some time ago, the writer found that most Ghanaian grade 11 students did not know how to show depth in a sketch of a cuboid (rectangular prism), a deficiency which made the application of plane geometry and trigonometry to 3-dimensional problems virtually impossible. Even if there were no connection between perception and geometrical ability, such blatant representational failures as these would point up the need for remedial measures as part of a student's general education.

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[1] The author's previously unpublished research reported in this review was mostly carried out while he was a doctoral candidate at the Ohio State University (F. Joe Crosswhite, advisor). The studies were supported by an Educational Development Award from the Ministry of Overseas Development, London and a grant-in-aid from the Society for the Psychological Study of Social Issues. The cooperation of students, teachers and educational authorities in Jamaica and Columbus, Ohio is gratefully acknowledged.

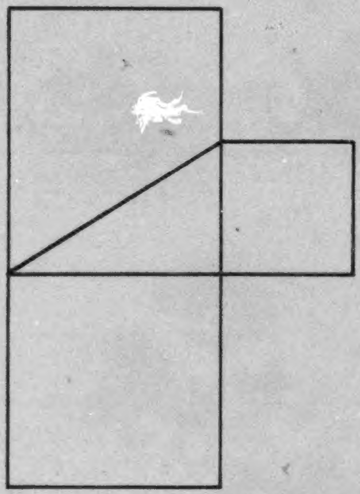

Figure 1. The above diagram was drawn by a Ugandan secondary school student who was instructed to construct squares on the three sides of the triangle (Skemp, 1971, p. 33).

Without apology for the poor coverage of strictly geometric concepts, this review will therefore be concentrated on cross-cultural research in related areas which are, or should be, of concern to anyone interested in the teaching or learning of geometry in developing countries. These areas cover the factors of intelligence known collectively as spatial ability, which I take to mean the ability to predict specified transformations of given geometric figures. More loosely, the abilities to be treated might be called "geometric intuition." Studies of figural reasoning (cognition of figural classes and relations in Guilford's, 1967, theory) and in the experimental psychology of perception (e.g., susceptibility to geometric illusions in Segall, Campbell, and Herskovits, 1966) will be omitted because these subjects have far lower face validity relative to geometry education.

Nonverbal Tests of Intelligence

Over the past fifty years, considerable effort has gone into the development of nonverbal intelligence tests. The major need has been to predict the relative achievement in education or employment of subjects who differ widely in reading ability, native language, or cultural background. Many of these tests, notably most of the performance tests developed for use in cultures where even paper and pencils are rare, appear to measure intuitive understanding of the geometric properties of 2-dimensional shapes, especially the way in which simple shapes fit together to make larger shapes. A test of this latter type is regarded as a good measure of general intelligence in a primitive population if it has a satisfactory reliability and a high loading on the first factor in factor analyses of batteries of similar tests (Ord, 1970). But one might equally regard such a test as a good measure of general spatial ability.

Before looking at some of the more relevant performance tests, it is mandatory to warn that cross-cultural comparisons are fraught with difficulties. There are so many reasons why mean scores should be different in different cultures (general levels of education, economy and health; familiarity with shapes used in the test and with the entire ethos of the testing situation, etc.) that it is never possible to deduce that populations differ in innate spatial ability, still less in innate intelligence. Comparisons between different unacculturated groups are safer than comparisons of Western and non-Western cultures, but still point to cultural differences before racial superiority. Also, one cannot be sure that a test which measures spatial ability in one population will necessarily do so in another (Irvine, 1965). To avoid a lengthy discussion of these problems, we shall treat each test at its face value and concentrate on qualitative rather than quantitative comparisons.

Block Design Tests

In the original block design test (Kohs, 1923), the subject is presented with a number of identical cubes each of which has four faces colored red, white, blue, and yellow and two divided diagonally and colored red/white and blue/yellow (Figure 2a). The test is to assemble a number of these cubes so that the top faces form a series of given designs (Figure 2c, 2d). The designs are all symmetrical and vary in size from 2 x 2 to 4 x 4; each design is either red/white or blue/yellow. The test was adapted for clinical use by Goldstein and Scheerer (1941) and as a subtest of general intelligence batteries (Wechsler, 1949; Matarazzo, 1972). A 2-dimensional version which employs red/white tiles (Figure 2b) has recently been developed by Ord (1970).

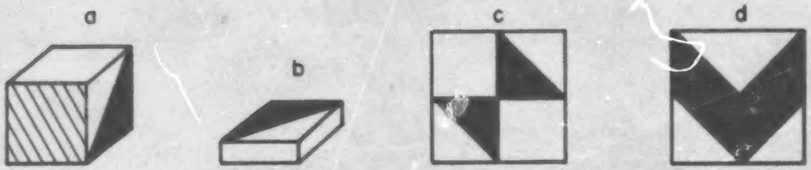

Figure 2. In block design tests, the subject is required to use cubes (a) or tiles (b) to copy designs such as (c) and (d).

The perceptual skills required for successful performance in a block design test may be demonstrated by describing the behavior of elementary school children as they attempt to copy the design in Figure 2 using tiles (Figure 2b). I have observed the following amongst both Jamaican and American students: Almost all students correctly select two white pieces and two "diagonal" pieces, and almost all show some difficulty in copying the orientation of the diagonal pieces. Amongst those who copy the design correctly, a few insert the diagonal pieces correctly the first time, without hesitation; some insert the pieces wrongly and then correct their orientation by trial and error; and some turn the pieces around and around in their hand to find the correct orientation before inserting them. Amongst students who fail this item, some try all orientations of the diagonal pieces but are unable to recognize the correct position; others simply insert the pieces at random, apparently unaware of the need to copy their orientation. Both the slow and the erratic efforts appear to reflect stages in the development of the capacity to deal successfully with oblique lines. In North American children, this development is marked by several achievements which suddenly appear at about age 6-7 years, e.g., the differentiation of left- and right-facing oblique lines (Rudel & Teuber, 1963), the formation of the concept of diagonal (Olson, 1970), and the more accurate copying of acute angles (Campbell, 1969). A large part of the variance in block design test score variance is thus attributable to differences in development of the perceptual concept of orientation.

Relatively poor performance on block design tests has been reported for several African samples. For example, Berry (1966) found that Temne (Sierra Leone) village and town adults scored significantly lower than comparative Eskimo and Scottish samples; Kellaghan (1968) found that Yoruba (Nigeria) village and city children aged about 11 years scored significantly lower than an Irish town sample; similar results have been obtained for 11-year-old schoolboys by Jahoda (1956), who compared Ghanaians with British farmworkers, and Vernon (1967), who comapred Ugandans with English schoolboys. Biesheuval (1949, 1952a) found several Zulu and Bantu samples significantly inferior to white South Africans of the same age. Some investigators (Deregowski, 1972b, McFie, 1954, 1961; Shapiro, 1960; Vernon, 1967) comment specifically on a greater tendency amongst Africans to rotate the entire pattern to a preferred orientation, for example, Figure 2d may be copied upside-down. Others (Dastoor & Emovon, 1972; Jahoda, 1956) have remarked on the difficulties which subjects have in copying the orientation of individual pieces. It is safe to deduce that Africans are on average significantly retarded in perceptual development as regards orientation. The dramatic effects this retardation can have on the learning of geometry have already been noted (see Figure 1).

There is another aspect of the pattern-copying task which has not been noted by psychologists but which is relevant to geometry teaching. Compare the designs in Figures 2c and 2d: In 2c, the "diagonal" pieces are isolated, whereas in 2d they fit together to make new shapes. Although orientation difficulties are still present in 2d, even subjects who solve 2c quickly have problems with the extra constructive requirements of 2d. The constructive element comes into most of the larger designs and could account for their greater difficulty. However, the construction of all such designs can be broken down into two problems: how to construct a large right triangle from two small ones (as in the upper and lower halves of 2d), and how to construct a parallelogram from two congruent triangles (as in the left and right halves of 2d). Often, the two problems must be solved simultaneously, for example, when fitting the last piece into the design in 2d. It would thus appear that part of the variance on block design test scores may be attributable to elementary geometrical knowledge (there must be many elementary mathematics textbooks that include the open-ended exercise, "What figures can you make by fitting together two congruent triangles?"). This is a further reason to regard the poor block design scores of most African samples as predictive of difficulties in their learning of geometry.

Outside Africa, Vernon (1965) found that 11-year-old Jamaican boys scored at about the same level as the Ugandan boys (Vernon, 1967) on the Kohs test. It was because of the low scores generally obtained by illiterates in Papua New Guinea on the 3-dimensional test that Ord (1970) developed his 2-dimensional version (Figure 2b); he also used this test with Australian Aborigines. Mitchelmore (1974) found that high-ability Jamaicans aged 7-15 years made no pattern rotation errors; their scores were significantly higher than average students of similar ages in Columbus, Ohio (Mitchelmore, 1975).

In startling comparison to previous results is the finding that scores not very different from those obtained by sophisticated European and North American samples are obtained by both literate and illiterate Canadian Eskimos and Indians (Berry, 1966, 1971; MacArthur, 1973; Vernon, 1966) and by Mexican Indians (McConnell, 1954). It may therefore be expected that Eskimos and American Indians will have special talents for geometry and other spatially-loaded pursuits (Kleinfeld, 1973).

Embedded Figures Tests

In the original embedded figures test or EFT (Gottschaldt, 1926), the subject was shown a simple figure for a certain length of time after which he had to locate and trace the figure embedded in a complex background (Figure 3). The test has been developed and used extensively by Witkin and others (Witkin, 1950; Witkin, Oltman, Cox, Ehrlichman, Hamm, & Ringler, 1973). The reason for the widespread use of this test is the finding (Witkin, Dyk, Faterson, Goodenough, & Karp, 1962) that scores are significantly correlated with performance on other perceptual disembedding tasks and with a more objective and self-reliant personality; persons high on these characteristics are said to be field independent.

Disembedding is clearly relevant to geometric problem-solving, where the first step is often to isolate appropriate figures (usually triangles) in a diagram of intersecting lines and curves--a step which Bright (1973) found was surprisingly difficult for elementary age school children. I have observed that students who do well on EFTs possess efficient strategies of searching for the distinctive corners and edges of a figure; this impression is strengthened by eye-movement studies (Conklin, Muir, & Boersma, 1968).

Embedded figures tests have been widely used in cross-cultural research, especially after Witkin's research on field independence became known. Again, relatively poor scores have been reported among African samples (Berry, 1966; Dawson, 1967a; Schwitzgebel, 1962; Vernon, 1967) and good scores among Eskimos (Berry, 1966; MacArthur, 1973; Vernon, 1966). These results would also tend to predict difficulties for African's learning of geometry, but not for Eskimos'.

In Jamaica, I have also observed that young students often respond without hesitation but draw a shape which is slightly different from the given figure (Figure 3). By contrast, older students prefer not to respond at all if they cannot locate the given figure. Reuning and Wortley (1973, pp. 48-49) report behavior similar to that of the younger Jamaicans amongst primitive Bushmen in South Africa. I suggest that EFTs measure conservation of shape in low-scoring subjects. Failure to conserve shape would also clearly limit geometric learning.

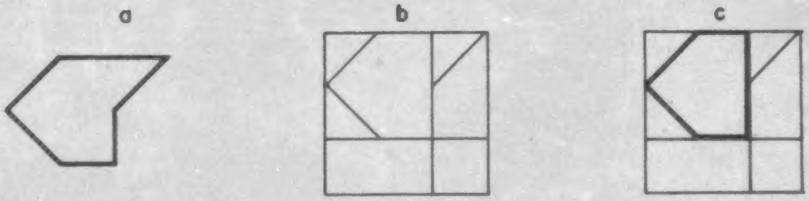

Figure 3. In a typical embedded figures test item, subjects are required to find a given shape such as (a) hidden in a complex pattern such as (b). Young Jamaican students were often found to copy the given shape incorrectly, as at (c). Copyright © Educational Test Service, 1963. Adapted and reproduced by permission.(see French, Ekstrom, & Price, 1963).

Other Tests

Several other performance tests of intelligence include elements of what I have called geometrical intuition. The most relevant of these are the various formboards, in which one or more geometrical shapes have to be fitted into a space to complete a larger shape. One series has been used extensively in South Africa (Biesheuval, 1949; Grant, 1970; Kendall, 1971) and another in Papua, New Guinea and in Australia (Ord, 1970), but they do not appear to have been used in cross-cultural studies.

Another interesting test is the Form Detection Test (Hector, 1964), in which subjects must find in a given array of dots sets of points which form the vertices of squares. Comparing Belgian Congolese illiterate workers with Belgian schoolchildren, Ombredane, Bertelson and Beniest-Noirot (1958) found that although the Africans were slower than the Belgians, their speed was not a function of task difficulty. The authors suggested that Africans' relative slowness therefore reflected cultural influences rather than innate differences in speed of mental processes.

In the Pattern Completion Test (Hector, 1958), three rectangles are given and a fourth must be placed to make a symmetrical pattern (mirror symmetry in half the items, rotational symmetry in the other half). Although illiterate Africans do not score as high as Europeans on this test (Fridjhon, 1961), they do not respond randomly even when the test task is demonstrated entirely in mime (Tekane, 1961). Interestingly, Tekane (1963) also found that, given a free choice, illiterate adults preferred a mirror-symmetrical completion whereas high school students used mirror and rotational symmetry equally often.

The Porteus Maze Test (Porteus, 1965) is another visual test which has been widely used in cross-cultural research. However, both its item type and the results of factor analyses show that whatever skills it measures are not directly related to geometry. Other tests which do measure relevant skills but which have not been used so extensively are reviewed by Ord (1970).

Piagetian Testing

There is now an extensive cross-cultural literature based on Piaget's model of intellectual development (Dasen, 1972). The usual finding is that children in non-Western cultures pass through somewhat similar stages as Western children, but at a slower rate and often without reaching the final stage at all. Retardation in the development of conservation of length, which would have obvious implications for the teaching of measurement, has been reported by Dasen (1974) and de Lemos (1969) amongst Australian Aborigines, by Isaacs (1975) and Vernon (1965) in Jamaica, by Okonji (1971) and Vernon (1967) in Uganda, by Page (1973) amongst Zulus in South Africa, by Prince (1968) in Papua, New Guinea, and by Vernon (1966) amongst Canadian Eskimos and Indians. In the same studies, retardation in conservation of area was reported by Prince and Vernon, but Goodnow (1962) found no retardation in unschooled Chinese of low socioeconomic status in Hong Kong.

There have been few cross-cultural studies of the development of geometrical and spatial concepts following Piaget and Inhelder (1956) and Piaget, Inhelder, and Szeminska (1960). Page (1971) reported the usual retardation for Zulu subjects in the use of coordinates for copying position, and Okonji (1971) found that very few Ugandan children aged 6-11 years used measurement in that task or in copying angles. Cowley and Murray (1962) studied the drawing of geometrical figures, haptic perception, the construction of the projective straight line, perspective, the coordination of perspectives, and the similarity of triangles and rectangles amongst Zulu and white South African school children. They found sequences of development similar to those described by Piaget with the usual retardation on the part of the Zulus. Dasen (1975) gave tests of linear and circular order, localization of topographical position, and representation of water levels to subsistence-level samples of Eskimos, Australian Aborgines, and Ebrié Africans in the Ivory Coast. His finding that Eskimos were superior to Aborigines and Aborigines superior to Africans on these tasks parallels the results obtained by Berry (1971) using block design, embedded figures, and other spatial tests. Once again, the conclusion is that African school children are likely to experience perceptual difficulties which could hinder their learning of elementary geometry.

Representation of water levels has also been studied by Dagnall (1970) in Papua, New Guinea and by Isaacs (1975) and Mitchelmore (1974) in Jamaica. The last-mentioned also studied the representation of the vertical. Before dismissing these representational skills as irrelevant to geometry, note that Mitchelmore found them highly correlated with other spatial skills and that Isaacs found only small correlations between them and conservation and school achievement scores.

Pictorial Depth Perception

Since Hudson (1960) found that Bantu primary school children had difficulties interpreting the depth dimension in a drawing of a 3-dimensional scene, pictorial depth perception has received considerable attention from cross-cultural psychologists. As excellent reviews have recently been published by Kennedy (1974) and Miller (1973), no details will be presented here. Findings for young children and illiterate adults in many developing countries all over the world may be summarized as follows:

1. Except in extremely isolated communities which have no experience of pictures, familiar objects can be identified fairly accurately from a simple line drawing or photograph (Deregowski, 1968b; Fonseca & Kearl, 1960; Holmes, 1963; Shaw, 1969; Spaulding, 1956).

2. Conventional signs such as those used to express movement are not very well understood (Duncan, Gourlay, & Hudson, 1973; Spaulding, 1956; Winter, 1963).

3. Cues used to represent depth in a 3-dimensional scene are poorly understood (Dawson, 1967a; Deregowski, 1968a; Hudson, 1967; Mundy-Castle, 1966; Shaw, 1969; Vernon, 1969).

4. The frequency of correct interpretation increases with age, education, urban influence, and cultural stimulation (Dawson, Young & Choi, 1974; Duncan et al., 1973; Holmes, 1963; Hudson, 1960; Kilbride & Robbins, 1968; Shaw, 1969; Sinha & Shukla, 1974).

5. Depth cues of size and superposition are the most easily interpreted, and perspective is the most difficult (Dawson, 1967a; Hudson, 1960; Kilbride & Robbins, 1968; Mundy-Castle, 1966; Shaw, 1969).

A child who is unable to "see the depth" in a drawing of a 3-dimensional scene is clearly going to have difficulties dealing with any school subject (e.g., solid geometry, science, geography) in which information is presented largely through "3-dimensional" pictures and diagrams. Although, as will be seen in the next section, secondary school students in developing countries have many difficulties making such drawings, the research literature cited above strongly suggests that they have few problems interpreting 3-dimensional diagrams drawn by others. However, the literature also warns elementary school teachers in developing countries that their children may well have some unexpected problems interpreting pictures, especially if they come from an impoverished background.

Pictorial Depth Representation

At the higher levels of technical and scientific education, it becomes necessary not only to interpret 3-dimensional pictures but also to draw diagrams to represent complex 3-dimensional configurations. To a lesser extent, the depiction of simple solid shapes is also required in many subjects in lower grades. Because drawing is so much more difficult than interpretation, the developmental process takes longer and its effects are much more visible. However, research on the drawing skills which are relevant to 3-dimensional geometry and the application of mathematics in physical space problems is rather sparse, both in the United States and cross-culturally. This topic will therefore be treated in greater detail.

The development of depth depiction in children's drawings of 3-dimensional scenes has been studied extensively by some art educators (Arnheim, 1954; Eisner, 1967; Eng, 1954; Lowenfeld & Brittain, 1966; Luquet, 1927; Munro, Lark-Horovitz, & Barnhart, 1942). Although there are differences in detail, four stages are generally reported in studies of Western children:

1. objects float in space, not properly related to each other or to any base line (age 4-7 years),

2. objects shown in correct topological relation to each other but without any depth depiction, often showing mixed viewpoints (age 6-10 years),

3. attempts to show depth by multiple base lines, overlapping and even size difference, from a single viewpoint (age 8-12 years), and

4. correct representation, objects related to a base plane, horizon in background (from about age 10 years).

Studies of single objects are much rarer. Kerr (1936), attempting to produce an intelligence test similar to the Draw-a-Man scale (Goodenough, 1926), found an increase in the proportion of children who drew a "solid" house, as opposed to a simple front view, from 10% at age 7 years to 60% at age 13-14 years. But only half of these drawings used oblique lines to show depth. Lewis (1962, 1963) studied drawings of a transparent sphere and a cubical house. For the houuse, she predicted five stages: (1) an isolated square face; (2) mixed viewpoints, no depth; (3) mixed viewpoints, some depth; (4) depth represented by drawing parallel sides parallel; and (5) depth represented by drawing parallel sides convergent. She found that the modal methods of representation were her Stage 1 in grades K to 3 and her Stage 4 in grades 7 and 8, being rather indistinct in grades 4 to 6.

Many of the spatial investigations of Piaget and Inhelder (1956) are also relevant to the study of pictorial depth representation, although the authors did not interpret their results in this way. They report stages in the drawing of a tree-lined road, poles on a hillside, and water levels in a tilted bottle. In all of these, an early stage involving the depiction of localized relations (perpendicularity), with consequent mixing of viewpoints, corresponds closely to the second stage of representation noted above. Stages in the drawing of a slanting circular disc and of receding railway lines illustrate the development of the perspective method of representing depth by foreshortening.

Several writers have noted that the preferred artistic style varies across cultures (Gombrich, 1960; Thouless, 1933). It seems likely that, in cultures where the prevalent style does not employ perspective, students would have extra difficulties learning to use the depth cues employed in the West (which are dictated by the central projection model of pictorial representation). Hudson (1962a) used this argument to explain why Indians in South Africa were less able than Europeans to pick up depth cues in line drawings of 3-dimensional scenes. Hudson (1962b) also found that illiterate African mineworkers drew cows, elephants, and cars using mixed viewpoints (Stage 2 above); they felt that a feature should be omitted from a drawing only if that feature was missing from the object depicted and not if it was merely hidden from view. The same principle is reported held by certain Australian Aborigines (McElroy, 1955). Deregowski (1969) obtained similar results when he asked Zambian servants and primary school children to draw a wire model of parts of a cube (the edges of the front and back faces, and one of the edges joining them); there was a clear tendency to draw the two square faces side by side instead of overlapping. In view of his further finding (Deregowski, 1970) that uneducated Zambian women actually preferred drawings which showed mixed viewpoints, it is difficult to say whether a person who makes such apparently primitive drawings is perceptually retarded or merely subject to a strong cultural influence.

I have seen only two cross-cultural studies which have examined the drawing of 3-dimensional scenes. Deregowski (1972a) estimated that rural Zambian children were 5-6 years behind English children in their drawings. Lester (1974) made a more careful comparative analysis. School children in Lagos, Nigeria and New York, New York, were presented with models of a simple domestic scene, which they then drew from three mutually perpendicular directions. Drawings were scored for overall spatial arrangement, for articulation of contiguous objects, and for consistency of viewpoint. There was a significant cross-cultural difference only for the first category, with Nigerian 10- to 11-year-old children scoring lower than American 6- to 7-year-olds.

None of the above studies gives very much information about the more specifically geometrical aspects of pictorial representation. I recently completed two surveys intended to fill this gap (Mitchelmore, 1974), which I shall now describe.

A Jamaican Developmental Study

Besides the work of Lewis (1962, 1963) cited above, only two previous investigations of the development of drawings of mathematical solids have been reported (Chetverukhin, 1971; Petitclerc, 1972). These papers include drawings of a cuboid, cube, and pyramid said to be typical at various ages, but give no supportive data. My first task was therefore to establish the developmental sequence and to develop a reliable test of a child's position in that sequence.

Eighty high-ability students (40 boys, 40 girls) aged 7-15 years in Kingston, Jamaica, were set individually to draw five small wooden models which were displayed at a fixed distance and orientation. The models were a cuboid, a cylinder, a pyramid, a cube, and a cone. Each solid was drawn twice, once after exposure for one second and once during an indefinite exposure. Stages obtained for the first four solids are illustrated in Figure 4; drawings of the cone were unscorable. The five stages of representation may be described as follows:

1. an outline of the solid or the face viewed orthogonally;

2. several faces shown but not in correct relation to each other, often both visible and invisible faces shown, usually no depth depiction;

3A. only visible faces shown, in correct relation to each other, but with poor depth depiction;

3B. all appropriate faces distorted in an attempt to show depth, but not correctly; and

4. correct drawing using parallel or slightly convergent lines to represent parallel edges of the solid.

The stages correspond fairly closely to those obtained for sketches of 3-dimensional scenes reported earlier, the third stage dividing clearly into two substages. Except for the cylinder, very few drawings combined depth depiction with mixed viewpoints (see Stage 3 of Lewis, 1963). The difference may be attributable to the fact that Lewis used a group administration procedure.

The validity of the test (called the Solid Representation Test) was established from several considerations. First, students rarely made more advanced drawings under the short exposure than under the long exposure condition (less than 10% of all drawings). Second, there was a fairly clear stage progression from grade to grade (Table 1). Third, a parametric analysis was made by assigning the five stages 1, 2, 3A, 3B and 4 scores of 0, 1, 2, 3 and 4 respectively; the reliability of the total score for the eight drawings was 0.96, showing a high consistency across solids, and correlations with block design and embedded figures test scores were 0.84 and 0.81, demonstrating satisfactory concurrent validity.

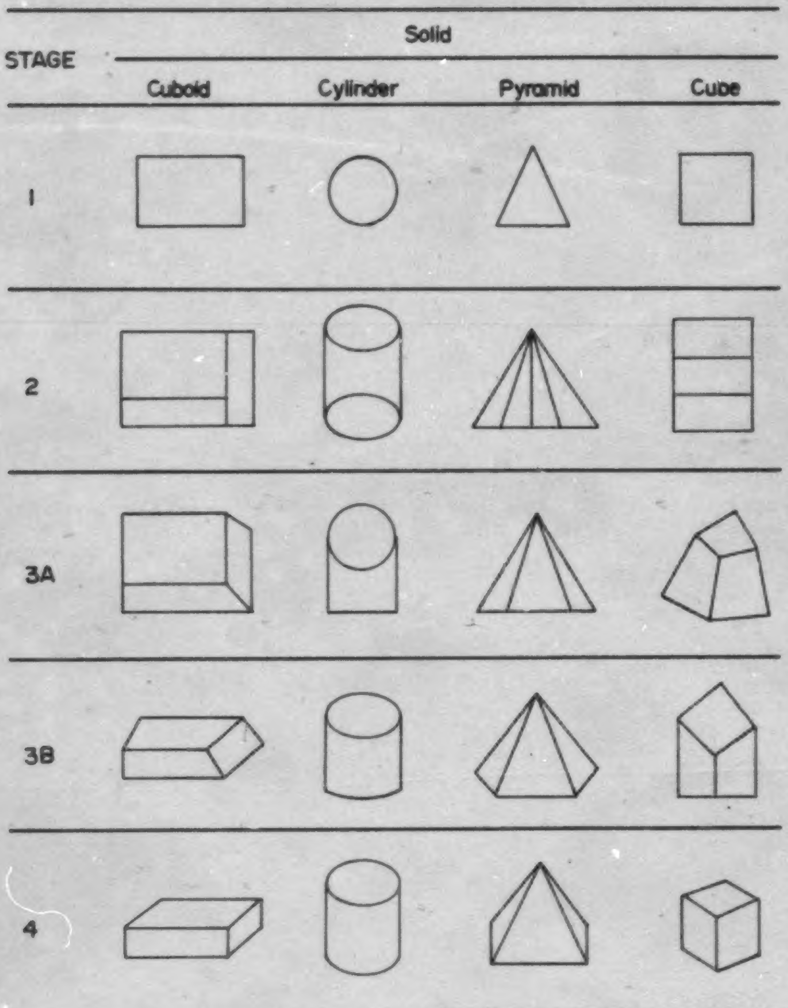
Solid
STAGE
Cuboid
Cylinder
Pyramid
Cube
1
2
3A
3B
4

Figure 4. Typical drawings at each stage of solid representation.

Table 1

Frequency of Stages on Solid Representation Test Combined Across Solids and Conditions in Jamaican Sample, by Grade

Stage	Grade				
	1	3	5	7	9
1	90	47	52	13	1
2	32	28	23	18	3
3A	6	26	19	28	21
3B		26	22	52	61
4		1	12	17	42

The Solid Representation Test was later administered to 64 average-ability students (32 boys and 32 girls) aged 9-15 years in Columbus, Ohio. The cone was replaced by a triangular prism, for which stages were predicted as shown in Figure 5. A high reliability (0.93) was again obtained, showing both that the sequence derived in Jamaica was also valid in Columbus and that the sequence was general enough to allow predictions to new solids. Table 2 shows the scores obtained by the two sets of students on the first four solids. The trend was highly linear across grades ($p < 0.001$), and the Jamaican scores were significantly higher than the American ($p < 0.05$). However, since the Jamaicans were chosen from high-ability classes and the U.S. students from average-ability classes, it is difficult to form any very definite conclusions from this result. A clearer comparison is available from a survey made using a group test of 3-dimensional drawing ability, to be described next.

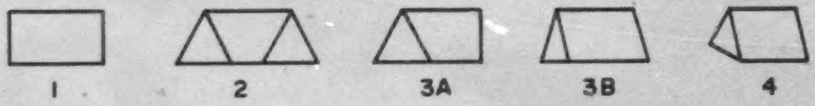

Figure 5. Typical responses predicted for each stage of development in drawings of a triangular pyramid.

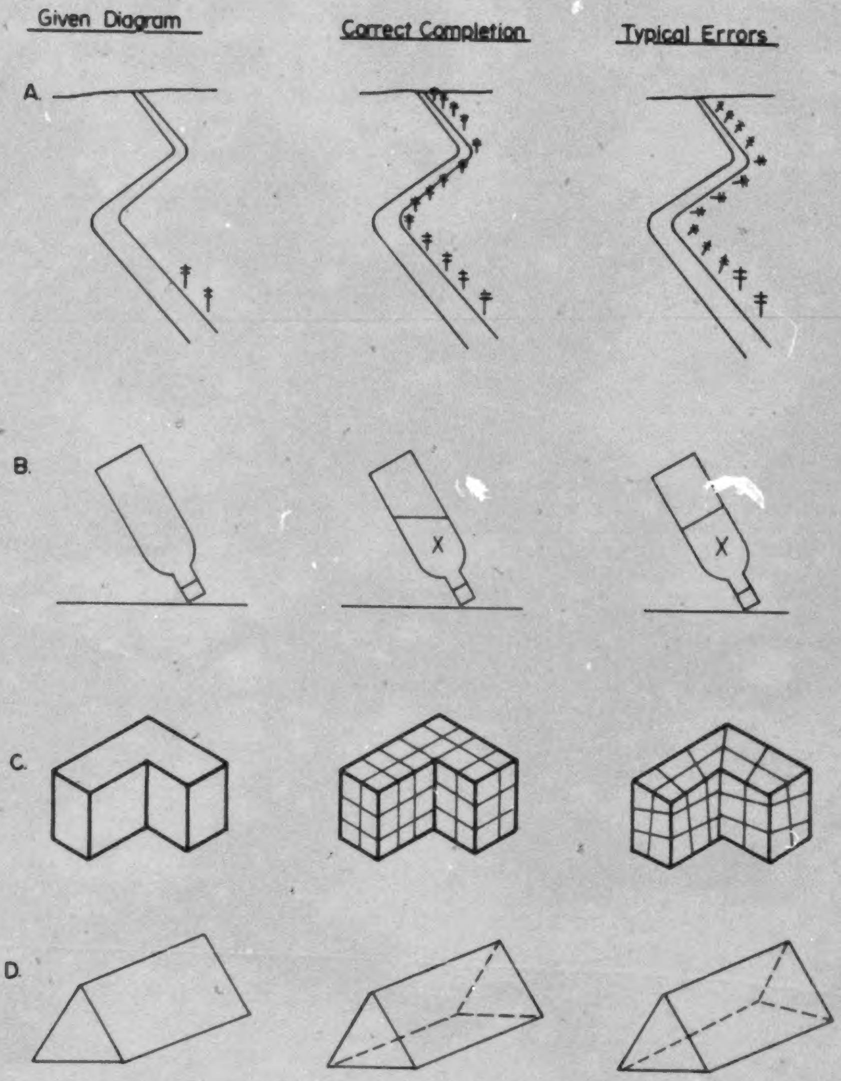
Given Diagram
Correct Completion
Typical Errors
A.
B.
X
X
C.
D.

Figure 6. Selected items, correct completions and typical errors from the Three-Dimensional Drawing Test.

Table 2

Results of Kingston and Columbus Students on Four Common Solids of Solid Representation Test, by Grade[a]

City		Grade			
		3	5	7	9
Kingston	Mean	1.29	1.37	2.33	3.09
	S.D.	0.92	1.21	0.92	0.43
Columbus	Mean	0.88	1.45	2.10	2.38
	S.D.	0.61	0.84	0.71	0.74

[a] N = 16 per cell; score is the mean stage of representation on a 0 - 4 scale.

A Jamaican School Survey

For my second survey, I developed a Three-Dimensional Drawing Test for group administration at the grade 9 level. (For provisos on the use of group tests in developing countries, see the next section of this paper). The following four exercises were selected after item analyses of pilot versions: draw telegraph poles alongside a road winding into the distance; draw the water level in four titled bottles; draw lines on the faces of four blocks to make each block look as if it was made from cubes stuck together; and draw the hidden edges of four blocks (Figure 6). The entire test was given in several different types of secondary schools in various regions of Jamaica, and the first two items were also given individually to the developmental sample mentioned above.

The four exercises all involve parallels and perpendiculars. To complete them successfully, the subject must draw parallels (or near-parallels) to represent parallels in space and ignore the perpendiculars. But as Piaget and Inhelder (1956) found, the youngest subjects drew perpendiculars to represent perpendiculars and ignored the parallels! The same tendency was responsible for most of the errors made by older subjects (see Figure 6). For example, in drawing an oblique view of a square network, lines still tended to be drawn perpendicular to the longer sides instead of parallel to the shorter sides. Even in subjects who apparently knew what to draw, there was a residual effect "pulling" the lines toward the perpendicular (Campbell, 1969). The reliability of this drawing test was estimated at 0.85.

The Three-Dimensional Drawing Test was also given in Columbus, Ohio to a sample of students from two average grade 9 classes. Again, the same scoring system proved applicable and there was only a slight drop in reliability (to 0.77). The scores of the Columbus and Jamaican students from two types of schools (the selective high schools, which are similar to British grammar schools, and the junior secondary schools, which are the nearest Jamaican equivalent to U.S. junior high schools) are presented in Table 3. Analysis of variance for the data in Table 3 showed highly significant main effects for sex and school type, and a non-significant interaction. (The sex difference will be discussed later). Post hoc tests by the Sheffé method showed that, for both sexes combined, the Columbus mean score was significantly different from the means for each type of Jamaican school ($p < 0.01$). The superiority of the Jamaican high school students confirms the result obtained using the Solid Representation Test and merely reflects the greater selectivity of Jamaican high schools. The comparison with the Jamaican junior secondary school students is more interesting. The junior secondary schools in Jamaica receive over 40% of the students who are not admitted to high schools, and their mean scores are very near to the estimated mean of all Jamaican grade 9 students (22.86 for boys, 17.69 for girls); they are therefore the nearest thing to "average" schools in Jamaica. It may be concluded that average grade 9 U.S. students are better at representing 3-dimensional geometrical relations than average Jamaicans in the same grade. A similar result was obtained in a preliminary study comparing West Indian and American samples of mathematics teachers (Mitchelmore, 1973).

Table 3

Three-Dimensional Drawing Test Results for Jamaican and Columbus Students, by Sex and School Type

Sex		Columbus	Jamaica	
		Junior high schools	High schools	Junior secondary schools
Boys	Mean	25.95	31.33	23.57
	S.D.	5.86	6.25	6.77
	N	21	54	54
Girls	Mean	22.33	24.59	17.15
	S.D.	6.22	6.36	6.35
	N	33	73	75

It may be said in passing that the Columbus students' performance on this test, though better than the "average" Jamaican's, was nothing to feel complacent about. In the first exercise (Figure 6a), even allowing for ±8° drawing error, only 2 out of the 54 students drew the telegraph poles upright alongside all three sections of the road. Only 8 drew water levels consistently within 6° of the horizontal (Figure 6b). In the item shown in Figure 6c, more than half the students made the error illustrated, and the item in Figure 6d was completed accurately by only 5. The relevance of these statistics to mathematics teachers is that students' drawing ability could probably be much improved by teaching one simple principle, namely that an acceptable drawing results when parallel lines in space are represented by parallel or nearly parallel lines on paper and other angles are deformed to maintain this invariance.

Paper and Pencil Tests of Spatial Ability

The problems experienced when Western paper-and-pencil tests are administered without modification to students in developing countries are well described by Schwarz (1963).In samples which are not test-sophisticated, "finding the correct answer may be no more of a challenge than finding the spot where it should be marked" (p. 675). Even when instructions are read in the native language, a large percentage of students do not understand what to do (Schwarz, 1961). It is therefore not surprising that there have been very few cross-cultural studies in which Western standardized tests of spatial ability have been used.

If one may assume that adults educated in the Western tradition will not differ greatly in test sophistication from one country to another, then two studies indicate that natives of developing tropical countries do not reach the same level of spatial ability as Europeans. Smith (1971) measured aspects of intelligence in 359 foreign students attending colleges of further education in England. He found significant differences in spatial and verbal test scores when students were divided into six groups by geographical area of origin: Students from Africa and the Caribbean ranked 5th and 6th in spatial ability, but 1st and 4th in verbal ability. Mitchelmore (1973) administered four of the National Longitudinal Studies of Mathematical Abilities (NLSMA) tests of spatial ability (Wilson, Cahen, & Begle, 1968a) to a group of mature West Indian mathematics teachers and a comparison group of student teachers at the Ohio State University. On all tests, the West Indians scored significantly lower than the U.S. students. However, the fact that the West Indian score on the first test taken (unfortunately, the only highly speeded test) was significantly lower than the mean reported for U.S. fifth-grade students (Wilson, Cahen, & Begle, 1968b), whereas the other three scores were significantly higher, throws doubt on the initial assumption of this paragraph.

Hendrikz (1973) gave several standardized tests, including the PMA Space Test, to African and European students in Forms I and III (approximately grades 7 and 9) of Rhodesian academic secondary schools. She found that the Europeans' mean score was only a little below the U.S. norm (47th percentile), but the Africans' was much lower (24th percentile). Hendrikz fails to take any account of differential test-sophistication between Europeans and Africans, who are educated in Rhodesia in completely separate systems below the university level. In this writer's view, her use of unadapted Western tests makes cross-cultural comparisons impossible.

In my pilot investigations in Jamaica, unselected grade 7 students faced with printed spatial tests showed the same signs of bewilderment which Schwarz (1963) reported. This finding throws doubt on the results of Vernon's extensive cross-cultural study (1969); about half of his tests were group tests, but he makes no mention of any special procedures used to ensure comprehension of the test task. Schwarz showed that it was possible to teach test tasks effectively using oral explanations, visual aids, and supervised practice (Schwarz, 1963; Schwarz & Krug, 1972). I used these techniques in Jamaica and never (well, almost never) found any students who did not know what to do when given the test paper. MacArthur (1973, 1975) also took care to include adequate practice and feedback in his group testing of Inuit Eskimo and Nsenga Zambian school children. Incidentally, both Vernon (1969) and MacArthur (1975) found that although Africans scored far lower than Europeans on the spatial tests, they scored near the European norms on the verbal-educational tests (cf. Smith, 1971), whereas Eskimos showed the opposite pattern, being near European norms on the spatial tests but far below them on the verbal-educational tests.

Schwarz (1963) notes an additional problem in adapting paper-and-pencil spatial visualization tests for African and similar populations: Subjects' difficulties with pictorial depth perception may contribute so much variance that the manipulative aspect of visualization is swamped. For example, in the usual type of surface development test, a different solid shape has to be imagined for each item. The I-D Boxes Test (Schwarz & Krug, 1972) reduces this variance by using the same shapes (two cubes) in all items and by supplying small wooden models of the cubes to each testee. I gave this test along with the Three-Dimensional Drawing Test to my Jamaican and American samples described in the last section and obtained very similar results (Mitchelmore, 1975).

Even with appropriate administrative modification, it is by no means certain that a test will measure the same thing in the new target population as it did in the country of origin. Irvine (1965) cites a case where an intended mechanical ability test apparently measured general reasoning. However, for the I-D Boxes Test, Schwarz (1964) reported validities of 0.30 - 0.47 for predicting achievement in the technical trades in Africa. In a sample of teacher's college men in Jamaica, I found a correlation of 0.34 with industrial arts grades. These validities are comparable with those obtained for surface development tests in the U. S. (Ghiselli, 1955). So all is not lost. Paper-and-pencil tests can still be useful in developing countries, provided they are treated with due care.

Spatial Training Programs

In view of the well-documented perceptual retardation of most African populations, and its probable effect on their learning of the spatial and mechanical elements of Western culture such as geometry and engineering which are seen as vital to economic development, it is surprising that very little research exists concerning the possibility of improving spatial ability by specially-designed training. What research there is, is just as ambiguous in its implications as the corresponding research carried out in the U.S. (For a summary of U.S. spatial training studies, see Mitchelmore, 1974.)

Two studies have dealt specifically with instruction in pictorial perception. Dawson (1967a) successfully taught 12 Sierra Leonean males aged about 18 to use depth cues in pictures. Meeting one hour a week for eight weeks, subjects were taught the standard depth cues by looking at outside scenery through a small hole, copying dominant lines on the window pane, and gradually learning to sketch directly onto paper. Their scores on Dawson's test of pictorial depth perception improved from 4% to 42% over a six-month period spanning the instruction, while a control group made only small gains. Serpell and Deregowski (1972) were less successful with Zambian grade 7 students. Three classes were given various treatments combining study of depth cues in photographs and films for four class periods, while a fourth class was a control group. Perhaps because of the short treatment period, many post hoc analyses had to be made before any significant difference was found. Duncan et al. (1973) report the successful use of similar methods, but cite no experimental verification.

A frequently cited study is that of McFie (1961), who reported significant gains on block design and other spatial test scores amongst a group of Ugandan technical school students between the beginning and the end of their 2½ year course. However, no control group was used, and the gains reported are well within the range of test-retest gains for control groups in other studies.

Several mathematics teachers in West Africa (McCrae, 1973; Mitchelmore, 1971) have suggested that secondary school students need many elementary practical-manipulative spatial experiences, apparently missing in most African homes, if they are to acquire a secure basis for later geometry teaching. One group incorporated many such activities into a "new math" textbook series (Mitchelmore & Raynor, 1967-75). Part of this latter program was recently subjected to experimental testing (Mitchelmore, 1974). Subjects were all 414 prospective elementary school teachers in a Jamaican college (194 men and 220 women), who were found to have a level of geometrical knowledge which was very similar to that of the Ghanaian first-year high school students for whom the course was originally designed. Six first-year classes and eight second-year classes were each randomly divided into two groups. The experimental groups studied a unit in which they designed, constructed, and sketched models of the basic solids, while the control groups studied a unit on statistics. Both units were individualized; first-year classes spent 4 weeks (16 periods) on each unit and second-year classes spent 3 weeks (12 periods) on each unit. Although there were highly significant differences on a unit achievement test, no significant differences could be found between experimental and control groups in posttest scores on the Three-Dimensional Drawing Test or the I-D Boxes Test. It seemed that students in the experimental groups learned mostly the names of shapes with which they were already familiar and learned little new about their spatial properties. Possible explanations for the lack of gains in spatial ability were the age of these students (mean 22.8 years) and their conservative attitude to individualized instruction.

Sex Differences

The well-known superiority of boys in Western cultures on tests of spatial ability (Buffery & Gray, 1972; Garai & Scheinfeld, 1968; Maccoby, 1967; Sherman, 1967; Tyler, 1965) has been replicated in many African countries, e.g., in Sierra Leone on block design and embedded figures tests (Berry, 1966), in Kenya on various performance tests (Monroe & Monroe, 1971; Nerlove, Monroe, & Monroe, 1971; Olson, 1970), and in Rhodesia on the PMA Space Test (Hendrikz, 1973). Stewart (1974) has recently provided an extensive review indicating the regular superiority of males over females in cross-cultural studies of psychological differentiation (especially when measured by block design and embedded fugures tests). By contrast, no sex differences in spatial ability have been found in Eskimos (Berry, 1966; MacArthur, 1967, 1973), Canadian Indians (Berry & Annis, 1974) and Australian Aborigines (Berry, 1971).

In my Jamaican studies described earlier I found males significantly superior to females on the group tests in grade 9 in all types of schools and in the teacher's college; the grade 9 difference was also found in the U.S. (see Table 3). On the individual tests, there was a peculiar developmental pattern in Jamaica which was not replicated in Ohio; it is illustrated in Figure 7 for the Solid Representation Test. I was so surprised at the pattern obtained in Jamaica that I tested a second sample of grade 5 students; the difference was still there. In Jamaican elementary schools, girls attend much more regularly and do better both in their regular school work and in the mainly verbal-numerical high school selection test taken in grades 5 and 6, so much so that in order to equalize the numbers of boys and girls admitted into high schools different pass marks have been used for the two sexes. This well-known situation makes the considerable superiority in spatial ability of the grade 5 boys even more impressive. The disappearance of this superiority by grade 7 could have been due to the fact that the older samples were drawn from high schools, where girls are probably of higher general intelligence than boys.

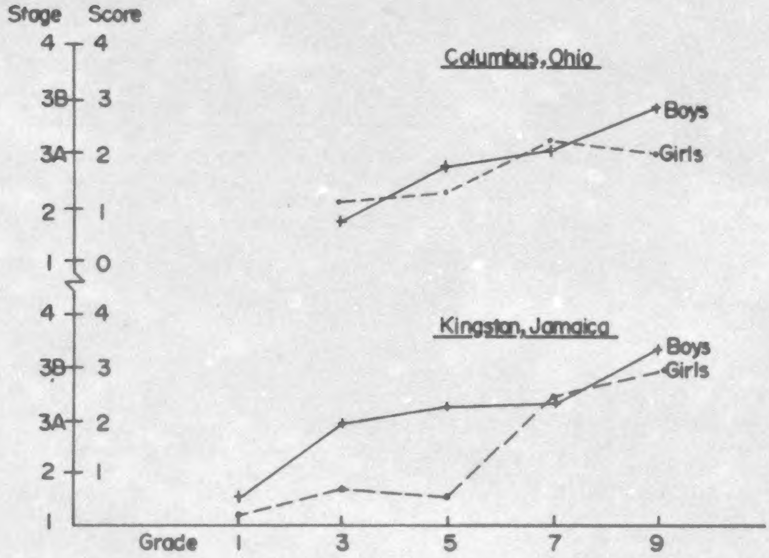

Figure 7. Results of Columbus and Kingston subjects on Solid Representation Test, by grade and sex.

There are two conclusions which can be drawn. First, if an elementary teacher in a country like Jamaica finds that the self-concept of the boys in her class is being harmed by the consistently better performance of the girls, she could counter this tendency by using more spatial manipulative-constructive activities in craft or mathematics lessons. Second, although girls may be lower than boys in average spatial ability, there are still many girls of high spatial ability and many boys of low spatial ability. In Jamaica, I found that, of the grade 9 students who scored over 50% on the I-D Boxes Test, 36% were girls; yet in schools where technical courses were offered, less than 2% were girls. It seems that many boys and girls (and not only in Jamaica!) are wasting their talents taking courses for which they are unsuited or not taking those for which they are suited.

Causes of Cross-cultural Differences

It is valuable to speculate on the causes of the cross-cultural differences reviewed above, since such speculation can point to factors influencing perceptual development in our own culture (whichever that may be). The big problem is that cross-cultural studies are inevitably correlational, so that there are always alternative explanations. For example, Eskimos may be superior in spatial ability to Africans because their environment requires highly developed navigational skills, because they are brought up to be more independent as children and are less specialized as adults, because their language contains more spatial terms and their art is more intricate, because their diet contains a greater proportion of protein, or because they have a lighter skin. Let us examine each of these possibilities in turn.

Physical Environment

That a person's interaction with the world outside his home affects his perceptual development is shown by two studies in which African farm children who travelled farther during their work and spare time tended to score higher on spatial tests (Monroe & Monroe, 1971; Nerlove et al., 1971). More recently, Berry and Annis (1974), comparing three Eskimo groups, found that the group which was most migratory had the highest block design scores. It is easy to understand why the need to navigate in apparently uniform snowfields (Kleinfeld, 1973) leads to increased perceptual differentiation, especially when compared to the sedentary agricultural existence of most African groups studied. Australian Aborigines, faced with navigational problems similar to those of Eskimos, appear to be intermediate in spatial ability (Berry, 1971; Dasen, 1975). As far as is known, no studies have been made of hunters who live in tropical rain forests, whose navigational skills may also be highly developed.

Some researchers have proposed a more passive environmental influence on perceptual development. Segall et al. (1966) showed that the amount of rectangularity (present in man-made objects) and openness in the visual environment were related to susceptibility to certain geometric illusions. Campbell (1969) suggested that Western students would be less inclined to tilt oblique lines towards the perpendicular (cf. Figure 6) because of the greater frequency with which they have viewed rectangular objects obliquely. Many have argued that the lack of pictures and manipulative toys in poor homes retards children's perceptual development (Biesheuval, 1943; Hudson, 1960; Vernon, 1969). My own view is that perceptual learning is an active process (Gibson, 1969) and that a poor environment is likely to be another result of the same social conditions which restrict the child's active exploration of his environment. Unfortunately, Vernon's hypothesis (1967, p. 341) that Africans are perceptually retarded by being bound to their mother's backs in infancy has not yet been tested.

Social Environment

Anthropologists (Barry, Bacon, & Child, 1957; Barry, Child, & Bacon, 1959; Berry, 1975; Cole, Gay, Glick, & Sharp, 1971) have found that certain types of economic activity are associated with particular practices in child-rearing, role specialization, and social stratification. At one end are migratory peoples, like the Eskimos, who acquire food by hunting; they do not store food and are constantly moving to new grounds. In this type of society, children are brought up to be self-reliant, there is little specialization of activities, and no elaborate authority structure. At the other end are the sedentary agricultural societies typical of traditional Africa, where food in plentiful supply at certain times must be stored to provide for the rest of the year. Here, the emphasis in child rearing is on responsibility and obediance, certain classes and both sexes have well-defined economic and social roles, and there is an elaborate structure of authority dedicated to the observation of precedent. There are several ways in which these social differences could cause differences in mean spatial ability.

Witkin et al. (1962) showed in Western samples that field independence (which includes high scores on embedded figures tests) was associated with an upbringing which encouraged independence of thought and action. It seems reasonable that a child who is encouraged to explore and find out things for himself instead of accepting the authority of adults may be expected to achieve greater perceptual differentiation and therefore spatial ability. The relation was confirmed in Sierra Leone by Dawson (1967a), who compared two adjacent tribes which differed markedly in child-rearing practices. Further evidence for the strong influence of the home is provided by the finding that sex differences in spatial ability are lower or nonexistent in cultures where both sexes are brought up to be independent (Stewart, 1974). Encouragement to be independent, objective and progressive would clearly be rarer in a society where roles and behavior are fixed by convention. Irvine (1969) has suggested that, if "intelligence" is the ability to get ahead in a given society, then it may be quite different in different societies. Amongst Eskimos, it would be physical and intellectual (planning and obtaining adequate food and shelter), but among African agriculturalists, it would be social and cultural (following the correct social practices). Therefore, we ought not to expect Africans to perform as well as Eskimos on spatial tests; their musical, dramatic and linguistic abilities are likely to be much more developed (Biesheuval, 1952b).

Cultural Influences

Several researchers have noted that African languages contain relatively few words for shapes or spatial relations (Cole et al., 1971; Gay & Cole, 1967; Ohuche, 1973; Olson, 1970; Stewart, 1971), whereas Eskimos have a rich spatial language (Berry, 1966; Kleinfeld, 1973; Whorf, 1956). The radical view that "we dissect nature along lines laid down by our native language" (Whorf, 1956) has been used by some to explain observed cross-cultural differences on spatial-perceptual tests (du Toit, 1966; Greenfield & Bruner, 1966; Littlejohn, 1963; Whorf, 1956). It is more consistent with the differentiation theory of perceptual learning (Gibson, 1969) to suppose that a people's language expands to match the perceptual differentiation which are important in their society (Deregowski, 1968a; Hudson, 1967; Olson, 1970). The same influences could cause the observed differences in artistic activities (Berry, 1966).

Cross-cultural studies have usually tried to equate samples for length of education. There is consistent evidence that experience of Western-style education is associated with greater perceptual ability (Dastoor & Emovon, 1972; Fonseca & Kearl, 1960; Gay & Cole, 1967; Kilbride & Robbins, 1968; Okonji, 1971; Olson, 1970; Shaw, 1969). It may be assumed that school attendance causes children to make differentiations required in Western culture but not in the native culture, for example while learning to read, write, interpret pictures and make drawings.

Nutrition

It has been reported (Dawson, 1971) that the daily food intake of the East African Kikuyu consists on average of 22 g fat, 390 g carbohydrate, and 100 g protein; for the Canadian Eskimo, the corresponding figures are 162 g, 59 g, and 377 g. However, although malnutrition is often associated with poor intellectual development, no causal relation has been demonstrated in humans (Vernon, 1969). Only Dawson (1967b) has shown any specific link between nutrition and perception. He found that a small group of Sierra Leonean adult males with a history of kwashiorkor (a disease resulting from protein deficiency) were significantly lower in spatial ability than their co-workers.

Skin Color

The possibility of a causal link between skin pigmentation and perceptual development was suggested by findings, in a U. S. sample, that the black children showed denser macular pigmentation in the fovea of the eye than the white children (Silvar & Pollack, 1967) and that dense macular pigmentation was associated with decreased contour sensitivity (Pollack, 1963). Smith (1971) gave another explanation. He noted the association in northern latitudes of lower levels of sunlight, less synthesis of vitamin D, and less absorption of calcium into the blood, and showed that the latter condition might promote vivid visual imagery.

Pollack's hypothesis was supported by Jahoda (1971), who found that the difference between the performance on a spatial task of equivalent Scottish and Ugandan samples was greater when the test was printed in blue (where the macular pigmentation shows the greatest absorbency) than when it was printed in red (where the absorbency is practically zero). However, Bone and Sparrock (1971) found no difference in the macular pigmentation of light- and dark-skinned West Indians. In my Jamaican study referred to earlier, correlations between skin color (rated on a 5-point scale) and scores on block design, embedded figures, and four drawing tests were all small and never approached significance. The relation between skin color and spatial ability must therefore remain an open question.

Needed Research

In the above review, I have described several ways in which students in developing countries differ in perceptual development from students in industrialized nations and suggested some reasons for those differences. One use of such cross-cultural studies is to warn visitors from the majority culture in Europe and North America (especially teachers of geometry, writers of textbooks, and constructors of aptitude tests) of possible obstacles to pictorial communication in the tropics (Selden, 1971) and of the intellectual strengths of certain apparently backward groups elsewhere (Kleinfeld, 1973). However, quite apart from the psychological questions left unanswered, much more research needs to be done on the educational implications of the known differences across cultures.

It is my feeling that geometry achievement is rather low in developing tropical countries. In Jamaica, the few geometry questions in the national Grade Nine Achievement Test have to be extremely simple, yet even so the most unlikely distractors are frequently chosen (Isaacs, 1974). As noted earlier, prospective elementary school teachers' knowledge of geometry is approximately equivalent to that of first year (grade 7) high school students. Geometry is infrequently taught outside the high schools, has only recently been introduced into the teacher's college curriculum, has little emphasis in the many public examinations, and is generally regarded with fear and suspicion by students and teachers alike. The small number of studies of geometry in developing countries may indicate that the subject is accorded similar low priority all over the third world. Yet geometry is important and should become more so as a country develops economically and requires a greater number of workers with minimal technical expertise. Teacher education is going to be particularly vital at such a time.

The standard of geometry achievement in developing countries should be investigated. The difficulties of conducting research in this area (sample selection, curriculum differences, language problems, testing procedures) are far greater than any so far tackled by the International Association for the Evaluation of Educational Achievement (Husen, 1974), but even a small-scale investigation could be useful. The Solid Representation Test shows high validity and reliability and could be very useful in uncovering difficulties in the 3-dimensional drawing which is so essential in many practical applications of geometry. It could now be used to test the oft-stated accusation that Africans are "unable to think in three dimensions" by comparing performance of Africans with that of Jamaican and American students.

At this point, I shall reveal my cross-cultural prejudices by predicting that English students would do better than corresponding American students on tests of geometrical and spatial ability. English and American schools tend to follow quite different approaches to geometry teaching. Geometry in England is rather informal. In elementary schools, the major emphasis is an active exploration of the visual environment and the study of spatial patterns; in secondary schools, many different systems (Euclidean, transformation, vectorial) are used to develop understanding of these patterns and the techniques for handling them. The use of manipulative materials in teaching elementary arithmetic probably contributes to perceptual development (Bishop, 1973), and diagrams and geometrical models are frequently used in other branches of mathematics at all levels. The emphasis is much more formal in American schools. In elementary schools, it is on abstract concepts almost exclusively in plane geometry; in secondary schools, on logical structure. Manipulative materials are rare (though becoming more common), and verbal and symbolic methods are used wherever diagrams can be avoided. My prime exhibit is the "Sesame Street" strip for January 5, 1975. The Cookie Monster calls pies, cakes and cookies "circles," when they are definitely 3-dimensional and nearer to cylinders. Moreover, the pies and cakes are poorly drawn, as Stage 3B in Figure 4. And why do The Arithmetic Teacher, The Mathematics Teacher, and most American mathematics textbooks, insist on using isometric drawings instead of the visually more pleasing (and more accurate) perspective method of representing rectangular objects? Why is it necessary to carry out sophisticated research to show that problems are significantly easier when accompanied by illustrative diagrams and significantly harder when the diagrams are misleading (Sherrill, 1973)? I hope someone will test more rigorously my hypothesis that Americans are underdeveloped in spatio-geometric concepts.

There is a desperate need for much more research into the development of geometric concepts. Such investigations should attempt to assess the involvement of spatial, verbal, and reasoning abilities in the learning of the selected concepts, to help find reasons why that concept is difficult and to suggest methods of overcoming those difficulties. For example, my studies of 3-dimensional drawing suggest that poor drawing stems from ignorance of the fact that parallels in space are best represented by parallels (or near-parallels) on paper. Instruction on this principle should lead to better drawings—at least amongst students who are young enough to learn new tricks. Some time ago, I noticed that Ghanaian grade 8 students had considerable difficulties with rotations and reflections; the fault of drawing reflections as shown in Figure 8 reminds one of Skemp's student (Figure 1), and once again illustrates difficulty with obliques. What sort of perceptual experiencs would help students overcome this difficuty? There is clearly much scope in developing countries for close clinical investigations into the development of a wide range of geometric concepts, the construction of theories of geometric concept formation, and the testing of instructional sequences intended to accelerate geometric concept formation.

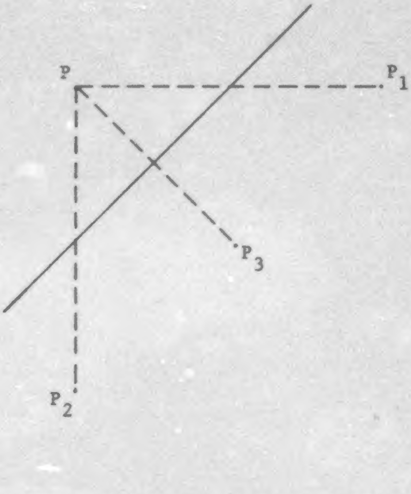

Figure 8. Ghanaian high school students have frequently been observed to draw the reflection of P at P_1 or P_2 instead of the correct P_3.

However, the greatest need is for the development of practical geometrical and spatial teaching programs and for their experimental testing. Most urgently required is the geometrical enrichment of the mathematics curricula in primary schools and teacher's colleges, previously almost exclusively arithmetical. For reasons just stated, I believe the English model would be more suitable for most developing countries; however, like everything else, this hypothesis is open to testing. Where the vernacular language has a limited geometrical vocabulary, it is possible that a program which emphasizes the names of concepts might be most effective, provided the concepts concerned shapes and relations in the environment and not diagrammatic abstractions. Also, in authoritarian societies, the encouragement of independence and self-reliance might pay dividends; attempts are being made in Jamaica to do this through the use of individualized instruction, although teachers' conservatism is likely to severely limit its application (Mitchelmore, 1974).

All along, it has been assumed that, in developing countries, greater geometrical and spatial ability is associated with better performance in the technical trades and professions, as it is in the industrialized nations. This assumption has a certain amount of empirical support, but clearly warrents further investigation. However, current test administration procedures are lengthy and require specially-trained testers. In the I-D Boxes Test, for example, the explanation and practice period is three times as long as the test itself. There is an urgent need for research aimed at producing a satisfactory format for tests of spatial aptitude which can be administered by school counselors. For if, as the great body of research reviewed in this paper indicates, the average level of spatial ability is lower in developing than in industrialized countries, it is doubly important for the future of industry and technology in the developing countries that students who are most capable of benefiting from technical education be reliably identified at the appropriate decision-points in the educational system.

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Transformation Geometry in Elementary School: Some Research Issues

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Some General Research Issues Concerning Elementary School Geometry

This paper consists of four parts: (a) an introductory section where some general issues concerning the development of mathematical concepts, in general, and geometric concepts, in particular, are discussed; (b) a section where some possible justifications for teaching geometry (in the elementary school), in general, and transformation geometry, in particular are discussed (research questions associated with each type of justification are also indicated); (c) a section where the use of indirect research procedures for investigating the kinds of questions that were suggested in part (b) are discussed; and (d) a section where ways that known mathematical systems can be used to guide the research procedures mentioned in part (c) are discussed. In particular, the question of whether (or in what sense) geometric concepts develop from topological to projective to Euclidean will be discussed in section (d). Throughout the paper, the emphasis is on transformation approaches to geometry.

The focus of this article will be on research having to do with children's concepts in transformation geometry. There are both practical and theoretical reasons for taking an interest in transformations. First, following the lead of secondary school textbooks, geometry chapters in elementary textbook series appear to be changing from a "traditional" to a "transformational" approach.[1] Nonetheless, in spite of the fact that there are some sound reasons for making this transition, recent studies by Kidder (1975), Williford (1972), and others, indicate that inherent difficulties may be involved in teaching "motion geometry" to children. But, what are some of the factors that contribute to the deceptive difficulty of many geometry concepts? A growing body of research (e.g., Freudenthal, 1973; van Hiele, 1959; van Hiele & van Hiele-Geldof, 1958) indicates that, especially in geometry, children make many mathematical judgments using qualitatively different methods than those typically used by adults. Yet, the nature of these differences is not clearly understood. Consequently, research concerning the evolution of spatial concepts would help mathematics educators better understand the difficulties that are implicitly involved in a wide range of mathematical concepts.

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[1] Following the advise of Weaver (1971), the Cambridge Conference (1963), the Ontario Institute (1967), and others, Burt Kaufman's Comprehensive School Mathematics Program project has developed "transformation geometry" materials for elementary school children. Addison Wesley, Silver Burdett, and Random House have also included transformation topics in their elementary textbook series.

A second reason for focusing on geometric notions in general is that most of the models (e.g., number lines, arrays of counters, fraction bars, Cuisenaire rods, etc.) and diagrams teachers use to illustrate arithmetic and number concepts presuppose an understanding of certain spatial concepts. Consequently, because of a lack of understanding of the spatial concepts, children sometimes experience misunderstandings about the models that are used. One of the most important goals of geometry research is to furnish information to devise "better" instructional models for teaching number concepts.

A third reason for investigating geometric concepts is to isolate some general principles for anticipating the relative difficulty of mathematical ideas. For example, if a child is operational (in the Piagetian sense) with regard to one concept, what does this imply about the child's ability to learn related ideas? If it is possible to find techniques to anticipate the relative difficulty of geometric concepts, then similar techniques may be able to be used to recognize the sequential presentation of arithmetic ideas--or instructional models leading to arithmetic concepts. For instance, van Hiele and van Hiele-Geldof (1958), Freudenthal (1973), and several Soviet psychologists (e.g., Sergeevich, 1971; Yakimanskaya, 1971) have explicitly described the way they believe geometric concepts evolve in children. Further, Piaget's theory suggests it may be possible to analyze, order, and equate concepts (and models) on the basis of their underlying operational structures (Lesh, 1975). Yet, basic controversies and gaping holes occur in each of these theoretical descriptions, and the controversies strike at the heart of many of the most basic issues in developmental psychology.

The working hypothesis underlying this paper is that spatial and geometric concepts make excellent areas of investigation for studying the development of mathematical concepts because many geometric concepts seem closely related to arithmetic concepts that are in the elementary school curriculum. Furthermore, children have usually not received explicit instruction concerning the geometric notions. Consequently, it is relatively easy to study the "natural" development of spatial concepts while minimizing the uncontrollable effects of specific prior training.

Piaget and Transformations

Although a variety of theoretical perspectives could be taken to develop the above themes (e.g., van Hiele, Freudenthal, Yakimanskaya, or Sergeevich), for the sake of continuity, comments in this paper will generally be made using Piagetian theory as a perspective. There are several reasons for adopting a Piagetian point of view in this paper. First, Piaget's theoretical perspective is more familiar to mathematics educators in the United States than several alternative points of view. Second, Professor Wirszup (in this monograph) develops some alternative theoretical perspectives. Third, Piaget's theory, more than any other, is currently being used (often unjustifiably) to support the adoption of a transformational approach to elementary school geometry, and, more generally, a "laboratory" form of instruction.

It is not odd that Piaget's name has been used to help justify the transformation geometry movement. Piaget's emphasis on cognitive activity, together with his claim that logical-mathematical operations are abstracted from interactions with concrete materials, seems to furnish natural justifications for a transformation approach to geometry. In fact, according to Piaget and Beth (1966), cognitive growth can be viewed as the child's gradual mastery of invariant properties under progressively more complex systems of transformations. Furthermore, Piaget's "Space" (Piaget & Inhelder, 1967), "Geometry" (Piaget, Inhelder, & Szeminska, 1970), and "Imagery" (Piaget & Inhelder, 1971) books have prompted some mathematics educators to claim that the "natural" development of spatial concepts in children may be analogous to the "transformational" development that is given in Klein's Erlanger Programm.[2] For instance, Inskeep (1968) has claimed "children understand topological concepts first, followed by projective, and finally Euclidean concepts" (p. 423). However, many of these kinds of generalizations are open to question in the opinion of most mathematics educators (e.g., Martin, 1976b; Steffe & Martin, 1974). Piaget (1973) himself has rejected many of the assertions that have claimed the support of his theory (see Furth, 1969).

Even though Piagetian theory has been used to help justify a transformation approach to the teaching of geometry, educators have tended to select only those aspects of the theory that support their preconceived biases, and have neglected other Piagetian points of view. Furthermore, many highly relevant aspects of Piagetian theory (which have been published under titles that do not reflect the mathematical nature of their content) have been ignored by mathematics educators. For instance, few mathematics educators are familiar with Piaget and Inhelder's "Imagery" (1971) book. While their "Space" and "Geometry" books are most relevant to the acquisition of spatial concepts, the "Imagery" book also contains many insights into the nature of children's ideas about transformations. On the other hand, at the same time that large portions of Piagetian theory have been ignored by mathematics educators, it has also become easy for people with widely divergent points of view to become self-appointed spokesmen for Piaget. In fact, the name "Piaget" has almost become a psychological counterpart of "Bourbaki" in mathematics--that is, a fictitious personality representing a whole group of people.

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[2] The Erlanger Programm (Klein, 1974) orders and compares geometries on the basis of invariance properties under various transformation groups; the theoretical construction proceeds from topological to projective to Euclidean.

Theory Building vs. Theory Borrowing

Discussions that took place at the "cognitive development" sessions of the most recent International Congress on Mathematical Education made it clear that there are often marked disagreements among scholars who have worked with Piaget (e.g., Papert, Remy Droz, etc.). In fact, there were clear disagreements among "Piagetians" who attended the research workshop in which this paper was presented. Such disagreements are healthy for a robust theoretical model. However, the trend in education has typically been to use cognitive psychology to help justify preconceived instructional biases, rather than to look at a theory in order to derive a consistent set of implications. Consequently, when a method of instruction is not effective in certain situations, the theory may be unjustifiably discredited (or rejected) rather than modified or extended to cope with the new difficulties. For example, if the name "Piaget" is replaced by "Dewey," most senior mathematics educators will be able to point out striking similarities between the "activity curriculum" movement of the 1920's and the "mathematics laboratory" movement of the 1970's. This cyclic history of curriculum change (i.e., enthusiastic adoption, followed by disillusionment, followed by rejection) indicates that theory building has not really been taken seriously by mathematics educators. For this reason, before going into the specific issues I wish to develop in this paper, it seems important to describe what I believe to be one of the major purposes of this research workshop. The objective has to do with "theory building" in mathematics education.

Perhaps it is unrealistic for mathematics educators to continue to search for "outside" theories that can be "lifted" and used without modification. Perhaps the emphasis should shift from "theory borrowing" to "theory building." One of the main benefits to be derived from theory building is that the theory seldom has to be completely rejected when conflicts are detected or when difficulties occur.

Theory building does not necessarily have to conjure up images of dull "ivory tower" activities that make no real difference anyway. For a beginning, theory building can simply involve organizing a point of view that can form a basis for communication among mathematics educators. In this way, individuals can profit by (and extend) the work of others. However, to avoid errors and inconsistencies, theory building inevitably attempts to describe the range of applicability of its major principles and to reconcile major conflicts within its point of view. Consequently, when difficulties arise, a theory should be more than a point of view that is simply accepted or rejected; it should be an explanatory and predictive "model" that can (and must) be gradually modified and reorganized to deal with progressively more complex situations.

While the history of science is filled with examples to illustrate the power of theory building, many mathematics educators would point out that mathematics education is more of a professional than a scientific discipline and that "the best practice of the best practitioners is still better than the best theories of the best theorists." However, this observation does not mean that theory building could not be helpful, it simply re-emphasize the point that theory building in mathematics education is in a very primitive state. Certainly no currently available psychological theories, including Piaget's, is ready for wholesale adoption by mathematics educators. In fact, it seems unlikely that a lasting theory, which can be adopted (without modification) by mathematics education, will ever become available. Even when a particular theory seems to be especially relevant to the acquisition of mathematical concepts, the mark of a useful theory is measured as much by the questions it generates as by the questions it answers. For this reason, every theory carries with it the seeds of its own destruction which soon require it to be modified and incorporated into a more comprehensive theory. But, continuous modification in mathematics education cannot take place by continuously borrowing theories from outside mathematics education.

The major objective of this series of research workshops is, in my opinion, to bring together people who are interested in theory building--not just people who are interested in collecting and analyzing data. Some questions that seem central to the concerns of this workshop are:

1. What is the mathematical status of geometric concepts studied by Piaget and other psychologists? For instance, Martin (1976a), Steffe and Martin (1974), and Kapadia (1974) have claimed that some of the "topological" relationships studied by Piaget (e.g., proximity, separation, order, continuity, etc.) are either not really topological properties at all (mathematically speaking) or else their mathematical meaning has been distorted. Deciphering the mathematical meaning of the concepts Piaget has studied is crucial for an accurate interpretation of his work as it relates to the acquisition of mathematical concepts. For example, Piaget's analysis of "continuity" seems to be especially questionable (see Taback, 1975).

Even though many of Piaget's tasks have been replicated and subjected to exhaustive psychological criticism (e.g., Dodwell, 1963; Laurendeau & Pinard, 1970; Lovell, 1959, 1961; Lovell, Healy, & Rowland, 1962), rarely have any of these replications questioned the validity of Piaget's mathematics. Furthermore, even though some Piagetian tasks have been almost over-replicated, other tasks have been largely ignored. For example, Piaget and Inhelder's "Imagery" book (1971) contains many tasks that are highly relevant to the acquisition of concepts pertaining to transformation geometry. However, these tasks have been neglected by mathematics educators, and so have many other "difficult to interpret" studies by Freudenthal, van Hiele, and various Soviet psychologists.

2. What is the developmental status of geometric concepts that have not been studied by Piaget or other psychologists? For example, to analyze the development of logical-mathematical thinking in children, Piaget has concentrated his efforts on children in the 5-7 and 10-12 year old ranges. Consequently, Piagetian research has focused on the cognitive processes used by first-graders (i.e., groupings) and sixth-graders (i.e., INRC groups), while neglecting children at intermediate grade levels. For this reason, and since Piaget has avoided mathematical ideas that are typically taught in school, it is only possible to make relatively crude inferences about how children's mathematical thinking gradually changes from a concrete operational mode of thinking to a formal operational mode. It is time for mathematics educators to forge ahead to investigate new concepts and new tasks that have not yet been considered by psychologists. For example, the Erlanger Programm seems to be a particularly pregnant source of uninvestigated concepts.

3. Can the Erlanger Programm or some other familiar mathematical system be used in some sense to model the development of children's geometric concepts? For example, Steffe (1973) has argued that certain mathematical structures may be able to describe the transitional phases through which elementary school children must pass in the development of geometric concepts, and he has suggested that these mathematical structures may be even better models of children's logical-mathematical thinking than Piaget's groupings or INRC groups. If Steffe's hypothesis is correct, this fact could be tremendously useful to mathematics educators who would like to construct curriculum materials which are consistent with the "natural" development of logical thinking in children.

4. In what sense is it true that geometric concepts develop from topological to projective to Euclidean? Martin (1976a), for instance, has analyzed some of Piaget's writings, and has concluded that Piaget's language in the area of space and geometry is not always consistent with common mathematical usage. Consequently, because of differences between mathematical and psychological terminology, it is not always legitimate to draw strong inferences from the psychological literature concerning the way mathematical concepts develop.

Justifications for Teaching Transformation Geometry

Even though this research workshop is primarily interested in organizing a theoretical model for research, it should not lose sight of pragmatic questions related to the development of instructional materials. In this section several types of justifications for teaching transformation geometry in elementary school will be considered, and examples of geometric topics that psychologists have tended to neglect will be given.[3]

Usiskin (1974) has given a number of reasons for adopting a transformation approach to high school geometry. First, he argues that a transformation approach is closer to the intuition of the student because it relies on simple symmetry arguments and on other familiar transformations. As evidence to support his claim, Usiskin points out that many proofs are significantly simplified when transformation techniques are used. Furthermore, he claims that evaluations of his own materials (Coxford & Usiskin, 1971), University of Illinois Committee on School Mathematics (UICSM) materials (1969), and materials that are used in many European schools (Jeger [German], 1966; Modenov & Parkhomenko [Russian], 1965, 1966; Troelstra, Habermann, Groot, & Bulens [Dutch], 1965; Yaglom [Russian], 1962), indicate that the transformation approach is especially well suited for slower students. In fact, the UICSM materials were written especially for slow eighth-graders.

Second, transformations can be used as a unifying theme in high school mathematics. For instance, most second year algebra concepts (e.g., real numbers, complex numbers, linear relations, systems of equations, algebraic systems, trigonometric functions, quadratic functions, logarithms, inverse functions, periodic functions, matrices, vectors, etc.) can be related to work in transformation geometry.

Third, transformation geometry is able to deal with a much wider variety of figures than more traditional approaches. For example a "Mira" construction instrument can be used to perform many constructions that cannot be accomplished using a compass and straight edge (Gillespie, 1973). Basically, the Mira's advantage is that it can perform rigid motions on many other figures besides polygons.

The major problem Usiskin identifies in teaching transformation geometry is that teachers are not as familiar with the approach. However, this problem is becoming less important because new teachers are taking college geometry courses which develop geometry using the "transformation" approach. Consequently, the desire to be "modern," plus the desire to present geometry in a way that is consistent with higher level courses, furnish sound reasons for high school geometry courses to convert from a "traditional" to a "transformational" approach. Nonetheless, the success of transformation geometry in high school does not necessarily mean that such an approach is appropriate for elementary school children. In fact, it is not even clear to many teachers that any kind of geometry should be taught in elementary school. Too often, the geometry chapters that are included in most elementary school textbooks are either skipped or skimmed over by teachers. One explanation is that teachers view geometry as a disjoint sequence of facts that does not lead anywhere but takes time away from their more important goals, namely teaching arithmetic.

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[3] J. N. Kapur the noted Indian mathematics educator, has given 27 reasons for including transformations in the elementary school curriculum (Kapur, 1970).

Elementary school is different than secondary school. Its students (who are called children in elementary school) are different; its classroom organization is usually different; its objectives are different. The goals of elementary school tend to be based on the assumption that time and effort should be focused on ideas and skills that are absolutely necessary for a minimally educated citizen to understand. Consequently, even if Usiskin's justifications are accepted as furnishing adequate rationale for teaching transformation geometry to high school students, his justifications are clearly insufficient to support (and were never intended to support) a similar conversion in elementary school.

Geometry (in general) is not considered to be important in its own right in the eyes of many elementary school teachers. Few teachers believe that studying geometry can contribute to the acquisition of arithmetic and number concepts or that geometry is a subject with a wealth of everyday applications. However, these attitudes are probably the result of the rather useless geometry topics that have been included in most elementary school textbooks and are not necessarily judgements about all geometry topics. In the past, the two most common reasons that elementary school teachers gave for teaching geometry were that many topics were fun or that they prepared children for high school geometry. In general, elementary school geometry has been only a "baby" version of high school topics. In spite of the recommendations of SMSG, the Cambridge Conference, and others, little consideration has been given to developing a different kind of "space experience" geometry for youngsters.

At least the following five criteria can be used to justify teaching a topic in elementary school: (a) The topic (e.g., addition, multiplication, fractions, etc.) may be considered to be important in its own right--without any "outside" justification. (b) The topic may contribute to, or reinforce, other important topics. (c) The topic may simply be fun and serve the function of luring children into enjoying mathematical problem solving experiences. (d) The topic can help to prepare children for higher level mathematics (e.g., high school geometry) (e) The topic can have important "real world" applications.

Each of the above five justifications will be evaluated in the following sections, and research questions associated with each issue will be considered.

Is Geometry Important in its Own Right?

Even though few elementary school teachers consider geometry to be important in its own right, geometry and spatial ability items constitute major portions of most intelligence tests (e.g., WISC, Stanford-Binet, PMA). Consequently, spatial concepts do seem to have some face validity as objects of study. Nonetheless, it is only in "special education" areas, like EMR (educable mentally retarded) or LD (learning disabilities), that many teachers have taken seriously the objective of providing experiences to cultivate the spatial abilities and geometric intuitions of children. However, the impact of most of these training activities is highly questionable. For example, in a study with normal K-2 children, Williford (1972) concluded that early training in transformation geometry may be expected to have little effect on the general spatial abilities of children. Shah (1969), on the other hand, seems to have shown that primary school children can learn transformations to some degree.. It should be noted, however, that the controls in Shah's experiment were minimal. More research is clearly needed to determine which (if any) spatial activities might be beneficial to young children, and what sort of transfer effects can be expected. Examples of some potentially beneficial types of geometry activities will be described in other sections of this paper.

Does Geometry Reinforce Other Important Topics?

Many elementary school teachers consider the geometry chapters of their texts to be isolated topics having little connection to other ideas in the book, and it is true that geometry topics are seldom selected because of their ability to contribute to other concepts. Yet, most of the instructional models[4] that are used to teach number ideas presuppose

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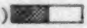
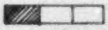
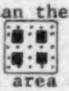
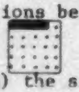
[4] For example, at grade levels when teachers are expected to teach fractions, many children become confused by the following types of questions.

1. Number line:
   How many points are on a line segment? Is it always possible to put a point between two given points?

2. Flannel Board: (a)  (b) 
   If a figure (a) represents one-half and figure (b) represents one third, how can the shaded portions be the same size?

3. Geoboard: (a)  (b) 
   Is the nonshaded area of figure (a) the same as the nonshaded area of figure (b)?

an understanding of certain geometric concepts which may not be understood. For example, in the upper grades, "area" or "volume" models and number lines are used to introduce fractions, and similar triangles may be used to illustrate proportions. Yet, there is abundant evidence (e.g., Gal'perin & Georgiev, 1969; Piaget & Inhelder 1967) that children frequently have problems understanding each of these models. Nonetheless, very little work has been done to isolate the geometric concepts that the models presuppose or to identify links between misunderstandings of models and misunderstandings of the ideas they are intended to illustrate (e.g., Lesh, 1976).

More than sixty years ago, Dewey and McLellan (1914) wrote a book whose central purpose was to outline ways that number concepts develop out of measurement operations involving basic geometric transformations. Similarly, Piaget (1965) and several Soviet psychologists (e.g., Gal'perin & Georgiev, 1969) have described ways that misunderstandings concerning number ideas are closely linked to a lack of understanding of certain geometric notions. For example, Piaget's number conservation task tests whether children realize that the number of objects in a set is invariant under simple spatial displacements (i.e., geometric transformations). Tasks such as these show that logical, arithmetic, and geometric notions are not initially learned as distinct categories of concepts for children. Rather, these three types of ideas exist in a confused and overlapping state for young children and only gradually become differentiated and coordinated. For example, young children tend to confuse judgements about: (a) the number of objects in an array of circles, (b) the density of the configuration, (c) the area covered by the array, and (d) the length of the rows or columns. Similarly, objects that are logically alike are often confused with objects that are spatially close together.

Piaget and Beth (1966) claim that logical, arithmetic, and geometric concepts each arise out of a common source, which is children's interactions with concrete materials. Consequently, because spatial experiences seem to dominate children's interactions with concrete materials, it would seem sensible to investigate the extent to which geometric experiences could facilitate or hinder the acquisition of arithmetic concepts. However, very little work has been done in this regard by mathematics educators in the United States.[5] Yet, Dewey's book alone furnishes a number of researchable hypotheses that could have significant practical payoff for arithmetic instruction in elementary schools.

For transformation geometry to reinforce other elementary school topics (i.e., arithmetic) as it reinforces secondary school algebra topics, it seems likely that the scope of the subject will have to be interpreted more broadly than it has been in high school. Transformation geometry is the study of invariance properties under systems of transformations, and the types of transformations that are admissable include more than rigid motions (i.e., slides, flips, turns), and similarities (i.e., stretches). For example, dissection theory concerns the study of invariance properties under subdivision and changed position transformations (see Figure 1). Perhaps dissection activities could contribute to children's understanding of area concepts and, in turn, contribute to their understanding of rational numbers (or fractions).

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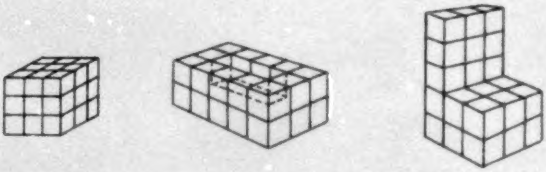
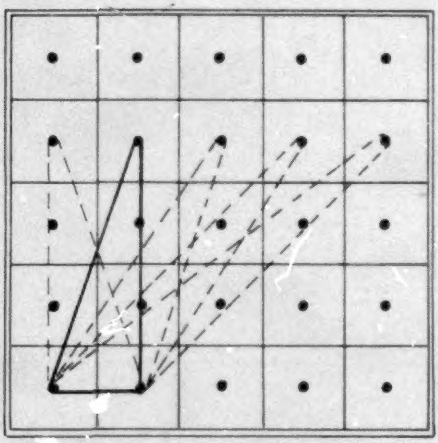
[5] A study analyzing figural models for rational number concepts was recently completed by Wagner (1975) at Northwestern University. A series of follow up studies is currently being conducted.

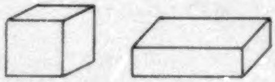


(a) Using soma cubes

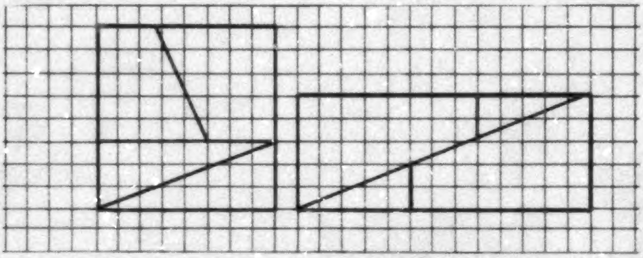


(b) Using a geoboard

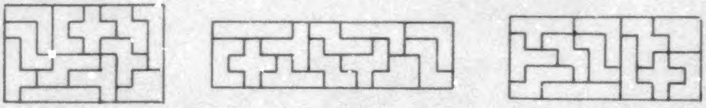
Figure 1. Transformations illustrating invariance properties.



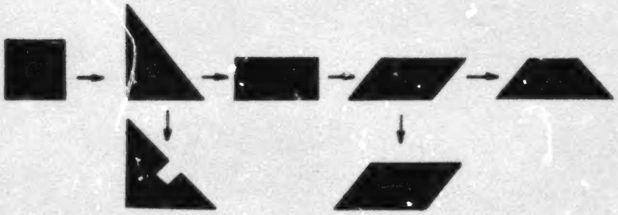
(c) Using sugar cubes



(d) Using graph paper



(e) Using pentominos



(f) Using tangrams

Figure 1. (continued)

Some instructional development efforts have been made in some programs (e.g., Developing Mathematical Processes, 1974; Comprehensive School Mathematics Programs, 1971) to devise measurement activities that contribute to children's understanding of arithmetic concepts. Nonetheless, geometry and measurement chapters of most elementary school texts still remain largely unconnected to the arithmetic and number concepts that are presented. Further, it is still not clear to most teachers how spatial-geometric (and measurement) concepts contribute to the acquisition of arithmetic concepts and number ideas.[6]

Does Transformation Geometry Offer Enjoyable Activities?

Because geometry chapters of elementary school texts often involve laboratory-type activities that children enjoy investigating, teachers sometimes use geometry as a break from usual classroom activities; because many geometry chapters are fun and can be studied independently, they can be used as extra activities for "smarties" or as a morale boost for "not-so-smarties." For example, each of the activities shown in Figure 2 has been shown to be unusually enjoyable for youngsters, and each can be related to the basic theme of investigating invariance properties under various geometric transformations.

Young children's facility with mirror cards, puzzles (e.g., tanagrams, soma cubes), tracing and drawing activities, and with "Mira" construction instruments indicate that many first- and second-grade children have already begun to acquire a number of transformation geometry concepts. But (a) how can these activities be used to gradually refine children's spatial concepts, and (b) how can such activities be used to lead into other important mathematics concepts?

Even in the case of popular geometry materials, such as geoboards, remarkably little has been done to analyze and exploit the geometric intuitions children have acquired. For instance, it is well known that children have difficulties dealing with coordinate systems like the ones needed to use rectangular graph paper. Yet, children often find it easy to construct and interpret some kinds of "treasure hunt" maps, and they often become quite skilled at geoboard games like "battle ship." Consequently, questions "a" and "b" become: (a') What geometric concepts are really needed to play games like "treasure hunt" and "battle ship"? Why are these games so much easier to understand than certain coordinate graphs? (b') How can games like "treasure hunt" and "battle ship" be used to gradually lead children to an understanding of rectangular coordinate graphs? How can primitive graphing concepts be used to facilitate the acquisition of arithmetic and number concepts?

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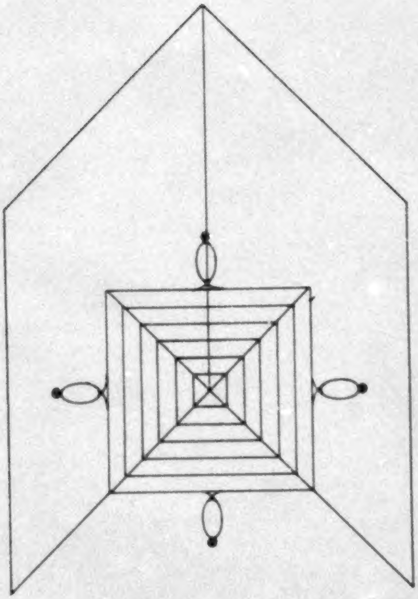
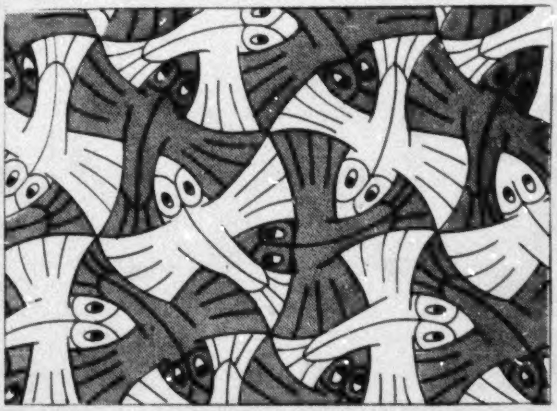
[6] Several of the papers in Lesh and Bradbard (1976) address this point, in particular, see Osborne (1976).

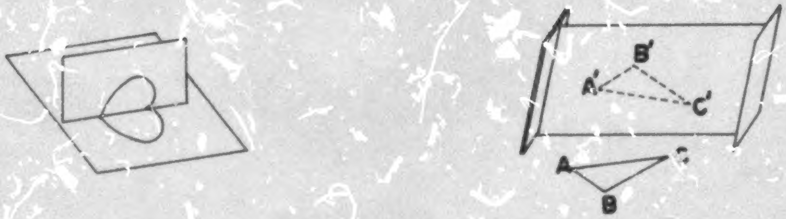


(e) Kaleidoscope (compositions)



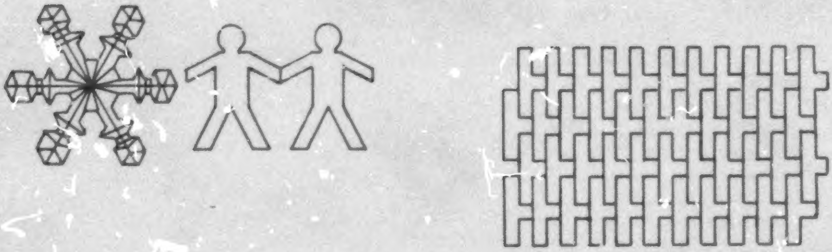
(f) Esher diagrams (compositions)

Figure 2 (continued)



(a) Mirrors (reflections)

(b) Mira's (reflections and constructions)



(c) Paper cutouts (reflections)

(d) Tiling (tessellations)

Figure 2. Enjoyable activities for investigating invariance under transformations.

The goal of studies dealing with questions like the ones mentioned above is to provide a response to critics who charge that laboratory activities tend to "make fun topics important rather than making important topics fun." The examples above, and the ones that will be mentioned in the following sections, seem to have the potential of being both fun and important. But, their potential must be exploited, and such exploitation requires further research to reveal the mathematical intuitions that are developed in children when they engage in enjoyable activities. Because many interesting activities seem to involve important mathematical ideas, and because interest often serves as an accurate rule of thumb to indicate readiness for dealing with an idea (Montessori, 1964; Smock, 1968; Standing, 1957), there is reason for optimism concerning the productivity of such research.

The "arrow diagrams" of the Papys (Papy, F. & Papy, G., 1970) furnish one example where simple graphs and mapping concepts have been used to teach arithmetic and number concepts to children. However, very little work has been done to determine what geometric concepts these mapping diagrams presuppose, or what misconceptions could arise if the geometric prerequisites have not been mastered. It is highly possible that graphs furnish a potentially powerful aid to the acquisition of number concepts, especially with regard to rational number ideas in the upper grades. Yet, mathematics educators have tended to avoid using graphs partly because it is difficult for adults to anticipate when their graphing illustrations involve ideas that children do not understand.

Two observations are striking concerning children's abilities to deal with spatial concepts. First, children are often able to perform tasks that seem quite intricate and complex; yet, many tasks that seem to be quite easy to adults turn out to be deceptively difficult for children to perform. For instance, one often finds kindergarteners surprising their teachers by being able to perform the following task:

> Put a strip of tape on the floor (e.g., |) saying, "Let's pretend this is a road." Put a chair on its side on one side of the road (e.g.,) saying, "Walk along the road, and put a chair on this side (indicating the side opposite the chair), so that you can touch the same thing here as here.

After giving an example of the desired response (e.g., -- the mirror image of the original chair), many kindergarteners can give solutions to similar tasks. Furthermore, some kindergarteners can give a correct response for configurations involving two or more chairs. In fact, some kindergarteners are even able to perform compositions of mirrow image transformations. For instance, if two "roads" are used, some kindergarteners can arrange the chairs as shown in Figure 3.

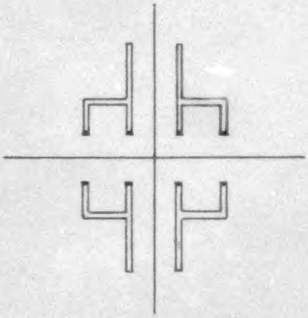
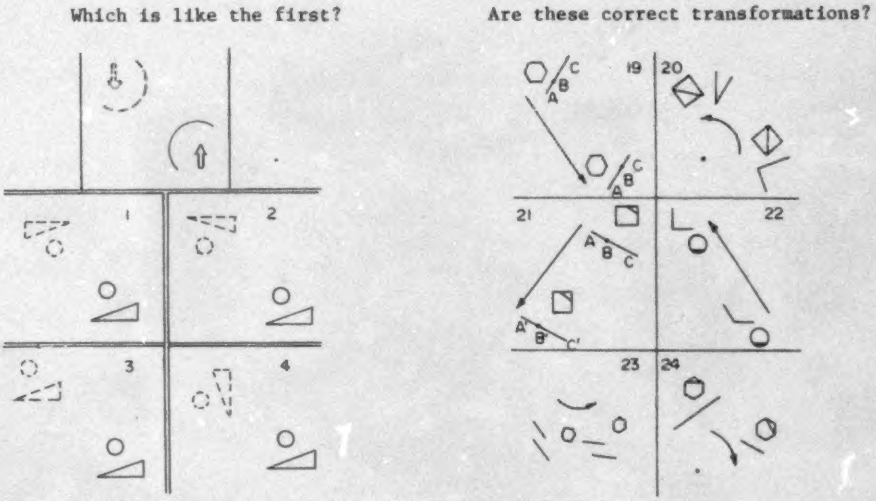


Figure 3. A composition of "roads" symmetry.

Modified versions of the above "chair" task can be given on a desk top by sliding, flipping, or turning paper cutouts and clear sheets of acetate. Tasks such as these illustrate that primary children have already developed some fairly sophisticated notions regarding geometric transformations. For instance, it is not difficult to teach many third graders to write letters on a window so that simple words can be read from the outside (e.g., ⅃ЯIᘓ--a reflection) or to teach them to write letters on a table top so that simple words can be read by a person sitting on the opposite side of the table (e.g., ⅂ꓤI⅁--a rotation). However, in spite of the apparent precociousness of the above abilities, Kidder (1976) demonstrated that sixth-graders have great difficulties with tasks like the ones in Figure 4.


Which is like the first?

Are these correct transformations?

Figure 4. Two tasks from Kidder's study (1976).

Because Kidder's tasks are very much like the UICSM "motion geometry" materials (1969), and because the UICSM materials seem to be the prototype for "transformation" geometry topics in elementary school textbooks, it is particularly important to conduct more research to determine which factors contribute to the difficulty of such tasks. Some of these difficult factors will be reviewed in the last section of this paper. For now, however, it is important to emphasize that the rules of organization children use to organize space are often very different from those used by high school students. Consequently, it is not always legitimate to simply offer elementary school children "baby" versions of high school or junior high school topics. Before junior high materials like those produced by UICSM are lifted wholesale (with only slight modifications) to be inserted in elementary school textbooks, more should be known about the geometric capabilities of youngsters. (Two current studies of this type are being conducted by Karen Shultz at Northwestern University and and Diane Thomas at Ohio State University).

Can Transformations Prepare for High School Geometry?

It seems likely that the transformation approach will eventually dominate high school geometry (Coxford, 1973); it already dominates college geometry. However, even if geometry is included in elementary school texts with the sole purpose of preparing children for high school geometry, it does not follow that elementary school children should be given concrete versions of high school topics. Perhaps some completely new topics would be more appropriate to develop the spatial intuitions of youngsters. For example, in the next section of this paper I will argue that projective geometry offers a particularly ripe area for instructional development. Few mathematical topics can compare with the simplicity, power, and elegant beauty of projective geometry, and few mathematical topics are so firmly rooted in concrete experience; yet, few laboratory activities have been developed to exploit the intuitive origins of projective geometry.

Often the decision to teach transformation geometry has meant that children will explicitly learn to deal with the basic rigid motions--slides (translations), flips (reflections), and turns (rotations). Yet, recent research at Northwestern (e.g., Moyer, 1974) indicates that young children do not think of transformations as continuous "motions" connecting two fixed states. Rather, children tend to think of transformations as changes in certain properties of the end states of transformations. Piaget and Inhelder (1971) have stated the following:

> The data. . . show quite clearly that it is easier for the child to imagine the product than the process, i.e., the movement as a trajectory. (p. 160)
>
> When the subjects attempt to imagine and draw in detail the . . . transformation, they represent the end-product less well than when it is the main object of the test. (p. 172)
>
> Imaginal representation bears first and foremost on the product of the transformation rather than on its successive stages. The image of the end-product is even somewhat better where there is no attempt to imagine the transformation itself. (p. 173)

Children are often able to solve problems while being unable to explain the steps that were taken to reach the solution. Similarly, during the initial acquisition of transformation concepts, children are usually not explicitly aware of the systems of operations they are using. It is one thing to organize reality using operations, relations, and transformations, and it is quite another to become formally aware of these operations.

According to Piagetian theory, performing second order operations (i.e., operating on operations) is a capability ushering in the period of formal operational thinking. For example, an analysis of tasks like those in Figure 4 indicates that formal operational thinking (e.g., hypothetical-deductive, if then thinking) is implicitly involved. In fact, a major conclusion of Kidder's study (from which the items in Figure 4 were taken) was that his transformation tasks were not mastered until the age when formal operational thinking begins to evolve. Unfortunately, however, Kidder did not carry out a thorough analysis of the difficulty-causing factors in his tasks.

A child's initial mastery of systems of mathematical transformations is somewhat analogous to the acquisition of correct rules of grammar. Children commonly use perfectly correct rules of grammar long before they are explicitly aware of these rules, and formal analysis of grammatical rules does not improve their speech. The situation is similar to Mark Twain's yarn about the centipede who became instantly paralyzed when asked to explain how his legs moved.

The intuitive mastery of a system of transformations is analogous to the acquisition of an unconscious habit--what is at first a habitual pattern for using a system of transformations to achieve some end later becomes a program in the sense that various substitutes can be inserted without disturbing the overall act. Furthermore, forcing a child to become explicitly aware of the transformations he is using may only confuse the child. Formal awareness of continuous movements seems to occur by piecing together "still life" frames somewhat the way old time movies were made--the more frames, the better the motion picture.

Coxford and Usiskin (1971), Freudenthal (1973), and others have clearly demonstrated the unifying power of using a transformation approach in high school geometry, and projects like UICSM (1969) have shown that many transformation topics can be modified to be appropriate for junior high school. But, serious questions remain concerning the study of transformations in the elementary school. There is little evidence that children think of Euclidean transformations as "rigid motions"--much less as combinations of "slides," "flips," and "turns." In fact, the evidence that is available suggests that "motion geometry" (i.e., the study of movements connecting end states) may be very different from "transformation" geometry, at least for elementary school children. Developing effective instructional materials requires that more be known about concepts that are confusing to children.

Educators have tended to treat children as though the only ideas they understand are those that have been consciously isolated and named. Perhaps more emphasis needs to be devoted to investigations exploring the intuitive (i.e., nonformalized) acquisition of systems of mathematical operations, relations, and transformations. There is a popular misconception that concrete and intuitive mathematics is inferior mathematics and that the viability of a mathematical topic is measured solely in terms of its formalization and abstractness. In fact, the situation is often exactly the opposite. Mathematical activity on the frontiers of research bears little resemblance to the kind of formal "problem sets" and "proofs" that clutter most high school geometry books; it bears a great deal more resemblance to activities that a youngster engages in when attempting to draw "three dimensional" pictures for a simple stereoscope (Figure 5). Trying to sketch a picture for the right eye that will coordinate with a picture for the left eye to produce the illusion of a three dimensional figure can require a great deal of geometric thinking.

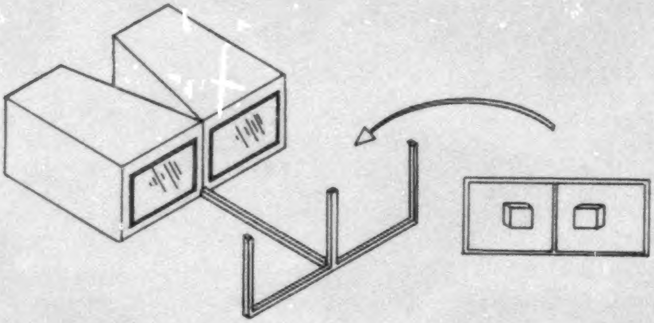


Figure 5. A simple stereoscope.

Premature formalization of subtle mathematical ideas can, in fact, mislead students into forming ideas that are not correct. For example, young high school students who repeatedly "prove" ideas that they consider to be obvious from experience often form false and distasteful impressions of the entire process of logical construction (Freudenthal, 1973). Yet, such students may be quite capable of formulating intricate and mathematically correct solutions to concrete problems. For instance, students who try to make paper models of polyhedral figures (see Figure 6) may discover insightful generalizations about interesting problems in combinatorial geometry. Furthermore, by attempting to isolate the mathematical essence of concrete situations, youngsters begin to appreciate the real beauty and power of mathematical modeling.

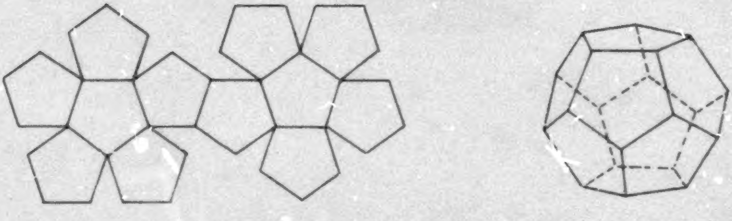


Figure 6

It is not the esoteric title of a mathematical topic, nor the concreteness of the context in which it is presented, that determine its mathematical viability. A topology course can be taught in a way that involves very little real mathematical thinking, or graduate students can be challenged by basic geoboard dissection problems. However, a great deal more work needs to be devoted to developing activities in which youngsters can use concrete materials to intuitively investigate mathematical ideas--and in which the concrete materials are not necessarily used only as a springboard to immediately soar to the highest possible level of abstraction (Trafton & LeBlanc, 1973).

Does Transformation Geometry Have Important "Real World" Applications?

Especially in geometry, it is difficult for adults to re-achieve preconceptual innocence so that they can empathize with confusions youngsters experience concerning many spatial concepts. For example, adults (and especially mathematics teachers) become so accustomed to conceiving the world as "three-dimensional," and they become so skilled at organizing their spatial experiences using rectangular coordinate systems, that they often feel as though they can "see" three mutually perpendicular axes built into nature. Furthermore, the "natural" feeling of rectangular coordinates persists in spite of the fact that they are often very awkward to use.[7] Adults realize that "the fourth dimension" (i.e., time) is not obviously a dimension like length, width, and height, but they forget that the "up-down" dimension was once considered to be essentially different than dimensions along the face of the earth.[8]

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[7] For example, points on the surface of the earth are usually located using a "polar" coordinate system involving reference lines of longitude and latitude together with measures of distance above or below sea level.

The difficulty adults have empathizing with children who fail to "see" coordinate systems is somewhat analogous to the experience people have when they finally see the disguised figure in a hidden picture puzzle. For example, look at Figure 7. Do you see an old woman? Do you see a young woman?

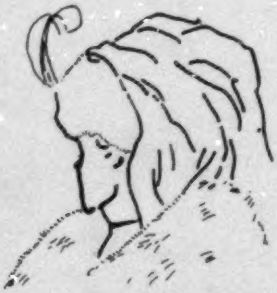

Figure 7. A hidden picture puzzle.

In much the same way that a hidden picture puzzle needs to be mentally organized for the picture to be seen, the real world needs to be organized for a three dimensional coordinate system to be seen. The question is, "What rules of organization must a child impose upon the world to read out the various coordinate systems?" Adults often deceive themselves concerning the nature of their geometric experiences. What they think they simply read out of the world, they, in fact, must read in.

To emphasize the distinction between information that can be read out, compared with information that must be read into the world, three observations about Figure 8 follow: (a) Figure 8 shows a hexagon made

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[8] It was once typical for height to be measured using different sorts of units than measures taken along the surface of the earth (Leach, 1973).

up of six equilateral triangles. (b) Figure 8 consists of seven dots connected by twelve line segments of equal length. (c) Figure 8 tessellates (i.e., a plane surface can be covered using nonoverlapping hexagons).

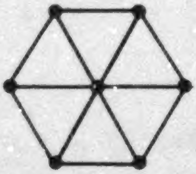


Figure 8. Hidden configurations.

Observation "c" probably seems more contrived and somewhat more sophisticated than observations "a" and "b." Observation "c" obviously requires the observer to read meaning into the figure, whereas less "reading in" seems necessary in the case of observations "a" and "b." However, according to Piagetian theory (Piaget & Beth, 1966), concepts like hexagon, twelve, and length also inherently require children to impose systems of organization on real world situations. In fact, according to Piaget, it is typical for mathematical concepts to implicitly require children to master systems of operations, relations, and transformations. Furthermore, the amount of information a person reads into (and consequently out of) a figure is limited by the systems of organization he is able to use.

Piaget has devoted a substantial portion of his mathematics-related research toward revealing systems of operations that are inherently involved in a wide range of mathematical concepts. However, less effort has been expended trying to determine why people sometimes do not use the organizational systems they are capable of using. For instance, look at the series of pictures in Figure 9, then reconsider Figure 8. What new meaning can be read into Figure 8?

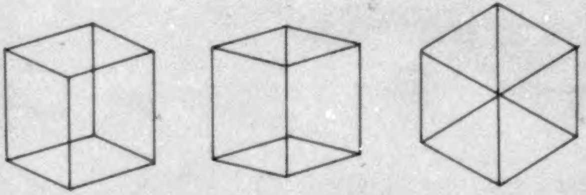


Figure 9. Another look at Figure 8.

The example in Figure 9 illustrates that having a system (or a concept) and knowing when to use it are two quite different matters; it also suggests that the problem of getting a person to use a system he has already mastered may be quite different than getting a youngster to make mathematical judgements involving systems he has not yet mastered.

Perhaps research investigating factors affecting a student's ability to use previously acquired concepts seems to be poaching in the domain of interest of problem solving theorists, but such issues are also important to educators who are interested in concept formation. For instance, according to Piagetian theory (Smock, 1973), the evolution of mathematical concepts typically involves both a figurative and an operative component (at least during early stages of development). However, the interplay between these two components of thought is by no means clear. It is well known that two tasks which are characterized by the same operational structure sometimes differ widely concerning degree of difficulty (e.g., décalages). However, factors contributing to these variations have not been thoroughly investigated (For a discussion of this issue, see the introduction to Laurendeau and Pinard's (1970) "Space" book). Nonetheless, it is known that situations are facilitating (or confusing) to the extent that there is some (is no) immediate connection between the figurative structure of the task and the operative structure of the concept involved.

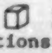
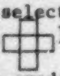
Dienes' (1969) "concrete embodiments" furnish examples where figurative models have been used to facilitate the acquisition of systems of mathematical operations. But many questions remain concerning the use of such models. For instance, to understand "regrouping," children sometimes find it helpful to work with the following types of materials: a counting frame (or abacus), building stacks, base 10 arithmetic blocks, base 4 arithmetic blocks, base 4 triangles, or money (e.g., pennies, nickles, and quarters). Furthermore, each of these materials is "good" in some sense and "not so good" in others. What materials are most abstract, or concrete, or complex? What materials will be easier for children to use? What materials will allow children to deal most directly with the concept of regrouping? What role does familiarity play in selecting models? What materials draw upon more intuitive notions that have typically been acquired by first- and second-graders? How many different materials should be used, and in what order should they be presented?

The questions in the preceding paragraph make it clear that even within the realm of geometric figures and "real world" materials, concrete-to-abstract and intuitive-to-formalized dimensions must be considered, and geometry seems to be a perfect context in which to study the relationship between figurative and operative aspects of thought. Piaget and Inhelder (1971) have stated:

> Spatial, geometric intuition is the only field in which the imaginal form and content are homogeneous. (p. 346)

> An image with logico-arithmetical content entails the conversion of non-spatio-temporal transformations into a necessarily spatial form. The spatial image, on the other hand, represents spatial content in forms that are likewise spatial . . . . The image of a number or a class is not in itself a number or class whereas the image of a square is approximately square. (p. 347)

However, even with regard to geometric concepts, there is never more than a partial isomorphism between the figural representation and the concept represented. For example, a picture of a cube must distort some properties of a cube in order to represent others.

Asking youngsters to draw a cube in perspective () illustrates the "simili-sensible"[9] character of figural representations. It also illustrates a striking difference between elementary school children and high school students. To pose the task, ask a child to put his nose on the edge of his desk. Then put a small (e.g., one inch) cube about six inches in front of him. Ask the child to look at the cube out of his right eye first ( ), then out of his left eye ( ), and then to describe differences that he sees. (Differences will be obvious if the sides of the cubes are painted different colors.) Finally, ask the child to draw (or to select from a set of predrawn pictures) a picture showing exactly what he sees with his right eye. The rather surprising result is that most fourth-graders will not be able to perform the task correctly, whereas ninth-graders will usually find the task quite easy. In fact, the picture that fourth-graders select (or draw) may not look anything at all like what they see. For instance, one commonly selected drawing looks like a box that has been cut apart and unfolded ().

The task above raises the question, "Why do fourth-graders and ninth-graders respond so differently to identical directions?" Fourth-graders' difficulties with the perspective drawing task cannot simply be attributed to a lack of drawing ability because: (a) pictures drawn tend to correspond to pictures that are selected from predrawn drawings (Piaget & Inhelder, 1967), (b) fourth-graders are perfectly capable of copying a picture of a cube.

An interesting observation about children who produce "unfolded-cube-drawings" on the perspective drawing task is that such children are often unable to give a correct drawing if they are asked to "draw what a paper cube will look like if it is cut along its edges and is unfolded." In the case of both the "perspective" and "unfolding" problems, children tend to draw what they understand, rather than what they "see" (Montangero, in this monograph; Piaget & Inhelder, 1971). That is, drawings are not so much like photographs as they are like symbolic representations.

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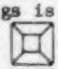
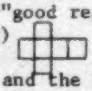
[9] The simili-sensible character of figural representations refers to the fact that some properties of a concept must be distorted to represent other properties (Piaget & Inhelder, 1971).

Adults become so accustomed to the usual "photograph-like" method of representing three-dimensional objects on two-dimensional surfaces that they forget that the method of using hazy background (and foregrounds) together with lines converging to a vanishing point (corresponding to the eye of the observer) was developed relatively late in the history of art. Early drawings tended to organize pictures using *conceptual* rather than *optical* relations (Bunim, 1940). For example, early Egyptian drawings combined several different points of view within one scene (e.g., heads were painted in profile, with eyes painted in front view). Further, size was used to represent the importance of the object rather than the actual relative size of the objects (e.g., kings looked like giants, while servants, animals, and inanimate objects were dwarfed). In fact, early Egyptian "medley of viewpoints" drawings closely resemble children's drawings in many respects. Just like many childrens drawings, the objects in some Egyptian pictures were drawn in a linear sequence, as though the figures were marching in a parade; objects that were conceptually related were drawn close together rather than objects that were actually (i.e., spatially) close.

Two-dimensional representations of three-dimensional objects must always distort some properties of objects to emphasize others. For instance, each of the following drawings is an equally "good representation of a cube: (a) (b)  (c) 

Figures (a) and (b) distort the measures of the angles and the length of the sides, and figure (c) distorts connectedness to represent the squareness of the faces.

Figure (a) is probably more common than figures (b) and (c), but it is no more accurate. In fact, for some purposes, figure (c) is best. For example, blueprints for houses (or directions for model airplanes) are more like figure (c). Because any representation must distort some properties to emphasize others, judgements about "betterness" are always dependent upon the function of the representation. But, many children have not learned to value the geometric properties their elders consider to be important. Consequently, children's judgements about the "goodness" of a representation may differ from those of adults. In the case of perspective drawings of a cube, children are usually more impressed by the squareness of the sides than they are by the measures of the angles or the way the sides fit together in three dimensions. Consequently, they are willing to distort properties they do not understand (and therefore do not consider important) in order to preserve properties they *do* understand (the squareness of the faces).

"Pseudo-conservations" furnish particularly striking examples of children's willingness to distort what they "see" in order to represent what they understand. For example, if a square is gradually tilted to look more and more like a very thin diamond (◆ ◆ ◆ —), many elementary school children are reluctant to draw nonsquares. For instance, some children's drawings look like progressively smaller squares (◆ ◆ ◆)--size being considered less important than shape.

Piaget & Inhelder (1967) have concluded that to perceive an object accurately, it is not necessary to be consciously aware of a point of view or of relationships between objects in the perceptual field. But to represent an object (or set of objects) in perspective requires a conscious awareness of the percipient's viewpoint, and of relationships between objects in the perceptual field. Children who have not mastered the system of relationships not only center on only the most obvious features of objects that they see, consequently failing to notice certain information that is available, but their representations also systematically distort features that may seem important to an adult in order to preserve features that are important to the child.

Intuitive activities in projective geometry seem to offer particularly ripe areas for instructional development. Some examples follow:

1. Children could be asked to arrange blocks so as to agree with the different viewpoints as illustrated in Figure 10 (Piaget & Inhelder, 1967).

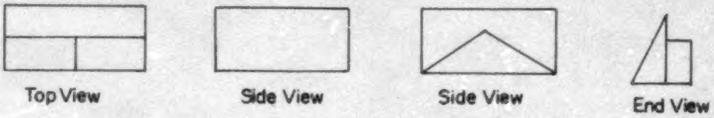


Figure 10. An activity relating to different perspectives.

2. Children could be asked to predict the shape of a cross-section made by different cuts of styrofoam models.

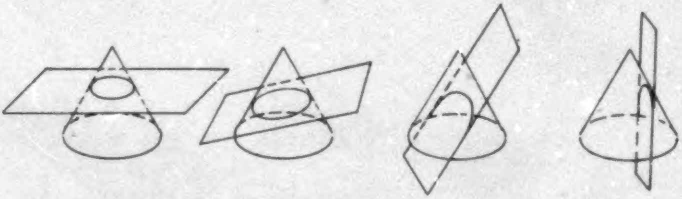


Figure 11. An activity relating to "sections."

3. Children could be asked to determine invariance properties under projections where the projections are constructed by using flashlights, sunlight, or an overhead projector. Specific questions might be phrased as: How does the shadow change when (a) the object moves, (b) the light source changes, or the screen moves (Dienes & Golding, 1967-1968)?

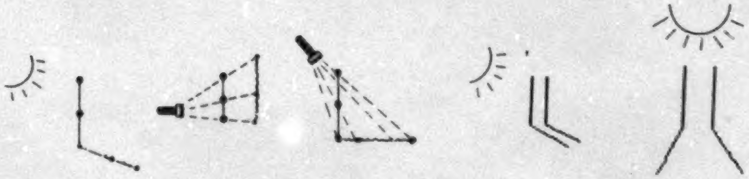


Figure 12. An activity relating to invariance properties under projections.

As far as real world applications are concerned, what could be more relevant and interesting than activities dealing with the everyday visual experiences of children? And, as far as mathematical viability is concerned, experiences with photographs, shadows, maps, and drawings can involve a level of mathematical thinking that even graduate students will find challenging. Furthermore, certain projections (i.e., similarity transformations) furnish figural models that could contribute to the acquisition of concepts concerning rational numbers, ratios, and proportions. In fact, it could prove wise to postpone the study of rational numbers until a variety of figural models have been investigated. However, for most concepts in transformation geometry, including projections, it is easy to pass from problems that are perfectly accessible to fourth-graders to problems that baffle many college students. Consequently, more research is needed to determine factors contributing to the difficulty of projective tasks, and more research is needed to investigate the processes that are involved for children to develop progressively more elaborate coordinate systems.[10]

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[10] A study concerning factors that contribute to the difficulty of projective concepts is currently being conducted by Naomi Fisher at Northwestern University, and a study concerning similarity transformations is being undertaken by Larry Martin at Missouri Southern State College.

Indirect Research Techniques

In the previous section of this paper, concerning possible justifications for teaching transformation geometry in the elementary school, a number of geometric concepts were mentioned that could give rise to important mathematical activities for children. However, in nearly every case, more information was needed about how children think about the ideas that were suggested.

The goal now is to examine a class of research techniques (i.e., indirect techniques) that seem to be particularly appropriate for investigating the nature of children's geometric concepts. Indirect research techniques are especially relevant to the concerns of this paper because they draw attention to possible similarities and differences between (a) the way a child organizes a set of mathematical concepts, (b) the way a mathematician organizes the set of concepts, and (c) the way teachers or textbooks organize the set of concepts. Indirect techniques are also of interest because they furnish a potentially powerful set of research tools that educators have frequently ignored, misused, or misinterpreted.

Notice that the goal of this section is not to deal directly with cognition in the sense of explaining how a child's mind works. Rather, the goal is to discuss the nature of children's geometric concepts and the progressive evolutionary stages through which these ideas pass. An important distinction is being made between (a) investigating a child's mind and (b) investigating the nature of childrens' concepts. The distinction is somewhat analogous to the difference between discovering how a computer works and describing the nature of the programs the computer is able to handle.

In the same way that a child may organize a set of mathematical ideas somewhat differently than a mathematician or a teacher, when a computer programmer begins to work with a new system, he may not be certain that the procedures he believes he has programmed correspond to the computer's interpretations of these procedures. Nonetheless, by feeding data into the terminal and observing the output, sound inferences can be drawn about the nature of the program as it is "internalized" by the computer. And, by investigating the effects of altering certain parts of the programs, important predictions can be made about prerequisite subprograms that must be developed before a more sophisticated program can be written. Furthermore, by investigating the nature of the programs that a computer is able to handle, indirect inferences can be made about the way the computer mechanism works.

In psychological research, there is an important distinction between research aimed at forming generalizations about the nature of children's concepts and research aimed at forming direct generalizations about cognition in children. For instance, concerning psychological theories that seem most relevant to mathematics educators, preconceived biases about the nature of mathematical knowledge appear to account for at least as much variance among theorists as preconceived biases about the nature of cognition. Nonetheless, few theorists have explicitly investigated the validity of their epistemological assumptions--even though the power of Piagetian theory alone makes it clear that espistemological research concerning the development of knowledge (and the nature of specific ideas) can furnish valuable information concerning the acquisition and creation of mathematical concepts.

One of the ingenious aspects of Piaget's theories is that they explicitly confront the fact that ideas (as well as children) develop, that is (a) a given idea may exist at many levels of sophistication, (b) the evolutionary development can be traced, and (c) the more primitive levels have seldom been investigated or accurately described.

Because of Piaget's strong interests in epistemological issues, his research has focused on investigating the nature of primitive conceptions of certain ideas. However, because few psychologists have shared Piaget's epistemological interests, his research has frequently been misinterpreted. First, the goal of epistemological research is seldom to furnish direct data about the way a child's mind works. Second, indirect research techniques seem to be particularly effective in epistemological research. Thus, from a psychologist's point of view, epistemological research is often indirect both in terms of its goal and its methodologies. For this reason, it is important to be aware of some of the peculiar strengths and weaknesses of indirect methodologies.

Twenty Questions

To illustrate some relevant points concerning indirect research methodology, imagine a group of people sitting around a table to play "twenty questions." The "leader" is designated by putting a "black box" on his head. The eye, mouth, nose, and ear holes are marked appropriately as input or output holes; words like "seven," "addition," "centimeter," or "triangle" are written on cards in a file labeled "mystery box #2." To begin the game, the leader selects and reads one of the cards from "mystery box #2," and inserts it into an input hole of his "black box." The group's task is to guess the idea written on the card by asking the leader questions. The leader's responses are limited to "yes," "no," or "uncertain."

After playing the game for awhile, players usually learn to avoid questions with "uncertain" responses and begin to develop optimal strategies to "focus in on" the mystery idea. More importantly, it becomes clear that the procedure the group is using is essentially a *negative* or indirect approach. That is, regardless of whether the responses are "yes" or "no," the net effect is to center in on what the idea *is* by gathering information about what the idea is *not*, thereby eliminating alternative possibilities. The only *positive* step in the process is made when one of the players actually attempts to guess the mystery idea. It is never possible to make a guess that is *completely* safe, but reasonably "safe" guesses can be made after the likely alternatives have been eliminated. In this way, it is possible to continue the game until the final guess can be made with considerable confidence.

Clinical Research

Piagetian clinical interview techniques use an indirect approach that is somewhat analogous to that used in "twenty questions." Twenty questions is not exactly analogous to the procedure used in clinical research, but it bears enought resemblance to serve as a basis for discussion. Certainly the questioning procedure used in "twenty questions" is not unscientific, even by the most rigid research standards; this is so in spite of the fact that "standardized questions," "large n's," and other common research standards are irrelevant. For the purposes of this article, the most obvious observation about "twenty questions" is that investigating the nature of a young child's conception of a praticular mathematical idea can involve two distinctly different types of research procedures: one positive (or direct), and one negative (or indirect). Further, some of the research standards that are appropriate for one of the procedures may be quite irrelevant for the other. In fact, each procedure involves distinctly different characteristic errors, and the important issue is to be very clear about which method is being used at a given time in a research project so that errors can be avoided.

As every mathematics student eventually learns, some problems that are very difficult to solve using a direct proof procedure become quite easy if an indirect proof procedure is used. Consequently, in solving mathematical problems, it is important to become adept at identifying the most appropriate type of procedure for particular questions. Similarly, researchers who become overly enamored with direct research techniques may be overlooking the power that is occasionally offered by an indirect approach. Because it is much easier to determine what an idea does not mean than to determine what it does mean, and because both types of information are crucial to thoroughly understand the nature of an idea, indirect research procedures can be of great assistance to educational and psychological researchers.

"Standardized" Questions

It is appropriate to note some obvious dissimilarities between a game of "twenty questions" and clinical interviews. First, the researcher usually has a broader goal in mind than to simply focus in on a particular child's conception of a given mathematical idea. Usually the goal is to make a general statement about the nature of a concept for a whole group of children. Consequently, if a researcher hopes to form generalizations about a group using data collected from a relatively small sample population, it is important to produce evidence that the sample accurately represents the group with respect to the relevant experimental variables. One method of accomplishing this goal is to randomly select the sample population and to use statistical arguments to justify the generalizability of results. However, statistical controls are not the only kinds that can be used, nor are they necessarily the best in all situations (e.g., direct experimental controls can be used in many instances).

Researchers must avoid techniques that are inconsistent with the theoretical model they are hoping to develop. For example, to have a large sample of subjects usually means that each subject is asked the "same" question. However, a Piagetian researcher who attempts to use "standardized" questions is immediately confronted with the problem that the theory challenges the existence of standardized questions. One fact that Genevans have established beyond a doubt is that (especially at "critical" stages in the developmental process) if all of the children in a group are asked the same question, individual children will nonetheless interpret the question in radically different ways.

Educational researchers in the United States have typically confronted the problem of standardized questions in much the same way that a "twenty questions" player learns to avoid questions with "uncertain" responses by attempting to avoid any unnecessary ambiguity in the questions and by shifting from asking single questions to asking clusters of questions whose responses constitute a behavioral definition of the idea being investigated. Nonetheless, the central problem is not solved. If the questions follow a computer-like "branching program" technique to focus in on an idea (as in the game "twenty questions"), then standardization is sacraficed because not all subjects are asked the same questions. On the other hand, if each subject is asked the same set of questions, the nature of the idea being tested is less clear.

There are two inherent drawbacks to behavioral definitions, both of which stem from the fact they are direct (or positive) research techniques. First, a behavioral definition is just what its name implies -- a definition. The definition reflects the researchers' biases concerning the nature of the idea and does not necessarily correspond to the child's conception of the idea. Second, a behavioral definition seldom yields a true definition of mathematical concepts. Davis (1964) referred to this fact when he wrote:

> If one states a specific set of really explicit objectives. . . this list seems always to be significantly incomplete: it is always possible to meet all the stated requirements, without actually achieving what was really desired. (p. 247)

The point is not that behavioral definitions are "bad." In fact, they probably constitute a necessary component of direct research procedures. The point is that behavioral definitions run the risk of being formulated too soon in a research project. The time when a behavioral definition can be formulated with confidence is analogous to the time when the final guess is made in a game of "twenty questions" that is, only after knowledge from indirect research procedures has been exploited.

Indirect Questioning

Isolating a child's notion of a particular concept is similar to the process a police artist uses to construct a picture of a criminal. The artist begins with a crude approximation and gradually refines the drawing using feedback from a witness. In fact, computers can be taught to recognize faces (from a limited population) using a "twenty questions" technique based on the identification of salient features (Harmon, 1973). Even though it would be nearly impossible to identify faces using only a direct technique, computers can "focus in on" a particular identity using a relatively small number of indirect steps.

Fortunately, the number of alternative ways geometric ideas can be conceived seems to offer far fewer alternatives than the number of forms faces can take. Therefore, indirect techniques seem feasible in epistemological research. Nonetheless, there comes a time when some form of direct approach must be used. In "twenty questions," it is when the final guess is made; in police detection, it is when the artist's sketch is used to apprehend the criminal; and in computer recognition, it is when the identification is attempted.

In clinical research, the direct approach usually becomes necessary when generalizations are to be made about a group of subjects. But, what group should be considered? One alternative is to consider a group of children of a particular age range. But there is usually considerable variation among children who are the same age. In most situations, a better technique is to form generalizations about children who have attained some other concept (e.g., children who are "operational" with respect to conservation of length). In essence, this approach amounts to a "transfer of learning" task; that is, if the researcher has used indirect techniques to "focus on" a child's concept of a particular idea, then fairly confident guesses should be possible about the nature of other "related" concepts. To confirm the accuracy of these guesses, one can again use indirect procedures to verify hypotheses concerning the nature of the related concepts. This "transfer of learning" procedure is a positive (or direct) transfer technique. However, once again, indirect techniques can be used. That is, negative transfer can be investigated by demonstrating how misunderstandings about one concept are related to similar misunderstandings concerning related concepts. For instance, if a child has difficulty coordinating part-whole relationships on classification tasks, and if he has difficulty with ordering relationships on seriation tasks, then he may have difficulty on number conservation tasks--because number concepts involve seriated classes (Lesh, 1975). Or, if a child has difficulties on "subdivision" and "change of position" tasks, then he may be expected to experience difficulties with simple measurement tasks (James, 1975).

One of the major difficulties with "negative transfer of learning" studies is that the researcher must be able to generate sequences of "related" tasks. But, in general, what source of information can be used to generate hypotheses about possible related concepts, and what source of information can a clinical interviewer use to generate tasks to focus in on the mathematical concepts of young children? Certainly one of the hallmarks of Piaget's research is the seemingly endless source of clever tasks and questions he has devised to investigate the nature of children's concepts. And, one of the strengths of Piaget's research rests on the "negative transfer" data he has gathered concerning similar types of errors children make on tasks he claims are related. Few people, including teachers who work with children everyday, would have predicted that children would perform as they do on many Piagetian tasks. Thus, for Piaget to correctly predict these surprising sorts of behavior is truly an important fact supporting the validity of his theory. But, what method can researchers less clever than Piaget use to generate tasks and questions to investigate the nature of children's concepts? The next section of this paper will illustrate how known mathematical structures can be used to investigate the nature of primitive concepts of length.

Using Mathematical Structures

A second striking dissimilarity between clinical research and "twenty questions" is that the game begins with no knowledge about the mystery idea. In clinical research, the name of the concept to be investigated is known, and various mathematical definitions of the concept are usually available. The goal is to determine which (if any) of the alternative definitions correspond to the child's initial understanding of the concept. For example, a researcher interested in investigating the concept "length" might begin by selecting a simple mathematical definition to guide his investigations. For instance, one point of departure would be to use the following definition of a metric space.[11]

> A metric space is a nonempty set S (whose elements are called points) together with a real valued function defined on S x S such that for any points x, y, z, in S, the following four conditions are satisfied:

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[11] Definitions of length involving an understanding of vector would be immediately discarded as being too complex.

M1) $d(x,y) \geq 0$
M2) $d(x,y) = 0$ if, and only if, $x = y$
M3) $d(x,y) = d(y,x)$
M4) $d(x,z) \leq d(x,y) + d(y,z)$

Because most people are willing to admit that it is possible for a child to attain a primitive concept of length without making numerical judgements, the researcher would probably be willing to accept as valid a concept of length which did not involve real-valued length judgements. But, would a concept be admissable as "length" if it did not satisfy properties M1 - M4?

Problems concerning properties M1 and M2 do not seem to arise naturally in concrete situations, but it is not difficult to devise tasks in which properties M3 or M4 will be consistently and emphatically denied by children. Nonetheless, in other contexts, these children may be able to demonstrate their mastery of a primitive concept of length. For example, children who can easily select the longest crayon from a box may look at a path up a hill (see Figure 13) and conclude that the distance from "a" to "b" (i.e., the length of the path up the hill) is different from the distance from "b" to "a" (i.e., the length of the path down the hill).[12] This judgement clearly denies property M3. Similarly, the child may look at strips of ribbon like the ones in Figures 14 and 15, and in each case decide that the lower string is longer, thus denying property M4.

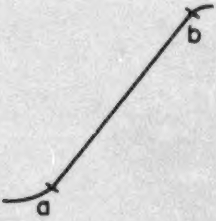

Figure 13.
Is $d(x,y) = d(y,x)$?

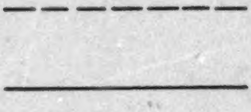

Figure 14.
Is $d(x,z) = d(x,y) + d(y,z)$?

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[12] A recent study investigating this phenomenon was conducted by Judith Musick (1975) at Northwestern University.

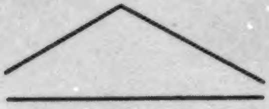

Figure 15. Is $d(x,z) \leq d(x,y) + d(y,z)$?

The examples in Figures 13, 14, and 15 illustrate the possibility of formulating a primitive definition of length which only applies to a restricted range of experiences that do not involve judgements concerning properties M1 - M4. The problem is to isolate the properties of a viable (but restricted) concept of length which does not involve properties M1 - M4. Observing children's behavior in tasks in which length judgements seem to be involved might suggest that the emphasis should be shifted from "length as a metric" to "equal in length" as a relation. Consequently, the following definitions of equivalence and ordering relations become relevant.

1. Equivalence: E ("is the same length as") is an equivalence relation on a set S if E is a set of ordered pairs of elements in S such that for any elements l, m, and n in S the following properties are satisfied:

E1) lEl.
E2) If lEm, then mEl.
E3) If lEm and mEn, then lEn.

2. Order: L ("is shorter than") is a relation (a strict partial ordering) on a set S if L is a set of ordered pairs of elements in S such that for any elements l, m, and n in S, the following properties are satisfied;

L1) It is not true that lLl.
L2) lLm, then it is not true that mLl.
L3) If lLm and mLn, then lLn.

Again, problems concerning properties E1 and L1 do not seem to arise naturally in concrete situations. But it is easy to devise tasks in which properties E2, E3, L2, and L3 will be consistently and emphatically denied by young children (James, 1975). For example, Piaget's famous conservation tasks furnish examples in which children will admit that a stick l "is the same length as" a stick m and that m "is the same length as" stick n but will deny that l "is the same length as" n. Similarly, other conservation tasks can be formulated in which each of the other properties is denied. The problem these tasks pose is to determine whether an even more elementary concept of length exists which does not involve the transitive and symmetric properties and which does not involve understanding that length is invariant under simple changes of position.

Following an indirect "clinical interview" procedure to focus on children's understanding of the concept of length, a number of famous psychologists (e.g., Dewey & McLellan, 1914; Gal'perin & Georgiev, 1969; Piaget et al., 1960) have formed a similar conclusion, that is, that the concept of length arises only after elementary systems of operations involving subdivision and change of position have been mastered. Furthermore, Piaget has concluded that subdivision and change of position operaions in turns arise out of even more basic "proximity-separation" relations analogous to the topological concept of "neighborhood."

Some quotes from Piaget, Inhelder, and Szeminska's "Geometry" book (1960) are:

> There can be no measurement, just as there can be no true representation of change of position, unless the space in which it takes place is structured by a system of references. p. 27)
>
> It is only by grouping relations or order and change of position simultaneously that children discover that objects which are moved leave behind them stationary 'sites'. This discovery leads them to conceive of space as a container or reference system which is independent of its content. (p. 80)

Several observations should be made concerning the clinical interview produces that were used to form the above conclusions: (a) The interviews involved an indirect method much like that used in "twenty questions." (b) Mathematical definitions could have been used (and probably were) to direct the inquiry. (c) The most basic systems of relations that children use to make mathematical judgements are globally analogous to the most general types of mathematical structures.

Mathematical Structures vs. Cognitive Structures

The close correspondence between basic "cognitive structures" and general "mathematical structures" should not be surprising because the methods used to isolate each type of structure is nearly the same. For example, in mathematics, a group of mathematicians publishing under the name "Bourbaki" isolated a small number of "matrix structures" which are fundamental to all of the various branches of mathematics because no one of them can be reduced to the others, and because all other mathematical structures can be derived from these by combination, differentation, or specialization. Regressive analysis isolated three basic types of structures which can be roughly characterized as follows:

1. Algebraic structures, the prototype of which is the group. These structures were distinguished in that their form of inverse operation was negation.

2. Ordering structures, the prototype of which is the lattice. These structures were distinguished in that their form of inverse operation was reciprocity.

3. Topological structures involving the concepts of neighborhood and connectedness.

Using a procedure similar to the method used by Bourbaki, developmental psychologists or mathematics educators can (a) analyze tasks that children perform, (b) characterize them on the basis of the systems of operations or relations they involve, and (c) order tasks on the basis of the complexity of these systems. In fact, using precisely this procedure, Piaget and Beth (1966, p. 186) have isolated three basic types of elementary cognitive structures (i.e., groupings) that are roughly equivalent to the three matrix structures determined by the Bourbaki group.[13] However, even though there is a global similarity among the most general types of mathematical structures and the most basic types of cognitive structures, there are also some striking dissimilarities.

Basic mathematical structures are simple (i.e., not complex) systems that apply in the widest possible variety of circumstances, and they are powerful (i.e., give rise to "nice" theories). However, because general mathematical structures must be combined and differentiated to be applied in most situations, they are also, in a certain sense, the least specialized and the least refined, and it is primarily in this sense that they are similar to basic cognitive structures. The first relations children learn to use to make mathematical judgements are simple and must be combined and differentiated to be applied in most situations, but they are also the least powerful because they are highly restricted, highly specialized, and closely tied to specific content. That is, basic cognitive structures are the most crude.

Grize (1960) has stated that it is possible for Piaget's "groupings" to be used to replace the matrix structures of Bourbaki. But, groupings, with their restricted combination rules and restricted associative laws, are "messy" structures that do not give rise to neat, tidy theories. For this reason, mathematicians would not take the trouble to formulate such awkward structures--especially as building blocks for a theory. For similar reasons, it seems unlikely that mathematicians will take the trouble to formulate most of the structures children use when they first come to master a given idea.

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[13] More recently, Piagetians have even found analogies of some basic ideas from category theory (Piaget, 1971, p. 26)

To model the thought processes of children, Steffe (1973) has suggested that Piaget's groupings can perhaps be replaced by more familiar mathematical structures. If Steffe's hypothesis is correct, it could be very helpful to mathematics educators who would like to construct curriculum materials which are consistent with the "natural" development of logical thinking in children. However, on the basis of the inherent differences that tend to occur between mathematical structures and cognitive structures, Steffe's point of view also runs the risk of attributing nonexistent processes to children's thinking.

It seems likely that mathematical descriptions of many of children's concepts will involve structures that mathematicians have not bothered to formalize. Nonetheless, for mathematics educators, the most important potential power of Piagetian theory will only be tapped if it is possible to establish links between the "natural" evolution of mathematical ideas and some mathematically viable organization of the concepts. In the meantime, mathematical structures can be used to effectively guide indirect research efforts aimed at isolating children's initial conceptions of various mathematical concepts; more importantly, indirect research procedures can be used to investigate the nature of concepts of intermediate levels of development which psychologists have tended to neglect.

Topological to Projective to Euclidean

To trace the evolution of logical-mathematical thinking in children, Piagetians have focused attention on the beginnings and endpoints of critical periods of development, e.g., the beginning of the period of concrete operations and the beginning of the period of formal operations. Logical inferences can then be made concerning intermediate levels of development. However, it is these intermediate levels of development that are of most interest to educators, and the inferences that can be drawn are crude at best. Consequently, more research is clearly needed.

In the previous section of this paper, a method of using mathematical structures to investigate cognitive structures was described. In this section the extent to which known mathematical structures can be used as models to describe the "natural" evolution of mathematical concepts will be discussed. In particular, the issue of whether geometric concepts develop from topological to projective to Euclidean will be considered.

From one point of view, it is a tautology to say that geometric concepts develop from topological to Euclidean. If the various geometric systems are defined in terms of invariance properties under groups of transformations, then Euclidean geometry is a more restrictive system than projective geometry, which is, in turn, more restrictive than topology. Consequently, two figures which are equivalent under Euclidean transformations are also equivalent under projective or topological transformations. Or, stated differently, Euclidean geometry is included in projective geometry which is included in topology (see Figure 16).

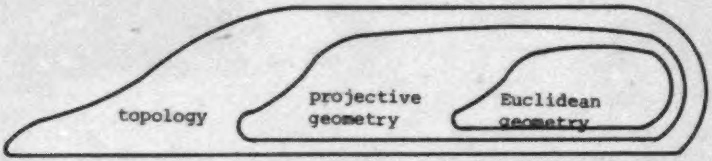

Figure 16. Topological concepts before projective and Euclidean.

According to Figure 16, there is no way it could fail to be true that spatial concepts develop from topological to Euclidean. However, as Figure 17 illustrates, this does not mean that all topological and projective concepts are mastered before any Euclidean concepts. In fact, Piaget has explained at length that projective and Euclidean concepts develop at the same time (rather than in succession as the above argument would suggest), and that some topological concepts develop relatively late. The following remark illustrates Piaget's position:

> The concepts of projective and Euclidean space develop together and are mutually interdependent. . . . The construction of physical reference frames, the final stage in the evolution of basic euclidean concepts, proceeded side by side with the general coordination of viewpoints, the salient feature of projective space. (Piaget & Inhelder, 1967, p. 419)

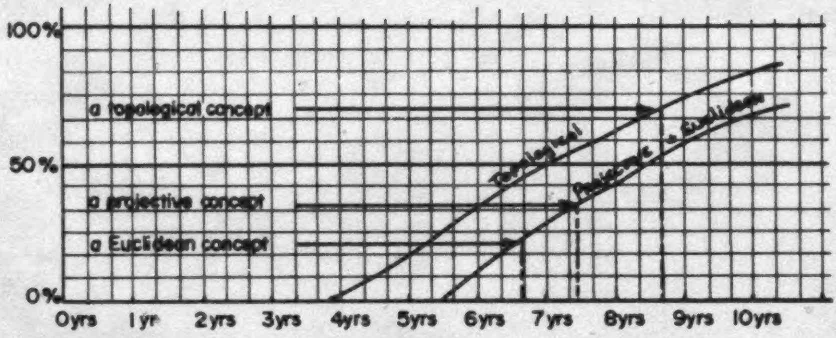

Figure 17. Variability within conceptual categories.

Even though the above problems do not pose a threat to Piagetian theory, other related problems are more troublesome. For example, the argument that topological concepts evolve before projective and Euclidean concepts relies heavily on the psychological viability of characterizing geometries in terms of invariance properties under *groups* of transformations. But, Piagetian theory claims that children are not capable of dealing with *groups* of transformations at the beginning of the period of concrete operations (Inhelder & Piaget, 1958). Furthermore, the progressively inclusive system of geometries illustrated in Figure 16 only holds if some conventions are adopted that seem psychologically artificial. For instance, the projection in Figure 18 does not preserve certain topological properties (e.g., continuity) unless "ideal" points are introduced. When line *l* is projected onto *l′* through the point *p*, the point *a* goes to *a′*, *b* goes to *b′*, *d* goes to *d′*, and *e* goes to *e′*. But, *c* becomes a vanishing point (i.e., it goes to a point at infinity), and a point at infinity goes to point *x*.

The example in Figure 18 illustrates that it is only in the projective plane[14] (or by making some other equally artificial convention) that projective geometry is a restriction of topology. But, do psychologists *really* want to imply that children's early geometry concepts involve an understanding of points at infinity? Do they really believe that children's topological concepts are consistent with the topology the Euclidean metric induces on the projective plane? Do they really mean to imply that similarities are mastered before congruences?

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[14] The projective plane is obtained by adding points at infinity to the usual Euclidean plane.

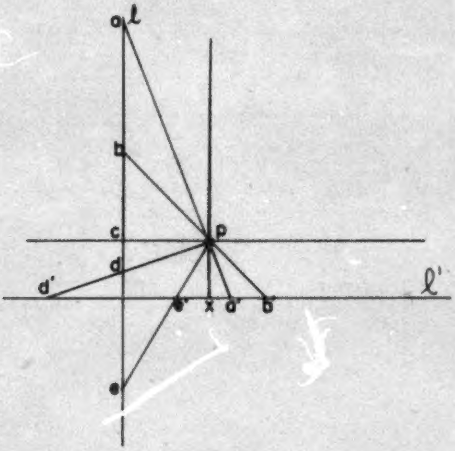


Figure 18. The projection of a line onto a line.

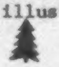
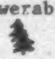
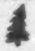
Questions like the ones in the preceding paragraph should not be dismissed too quickly as having obvious answers. For instance, certain psychologists (e.g., Piaget, Dewey) have suggested that primitive similarity concepts are implicit in children's early ideas about congruence, and it may not be foolish to investigate the possibility of a primitive concept of "points at infinity." Nonetheless, it appears that psychologists have used mathematical structures in somewhat the same way educators have used psychological theories. That is, they have referred to the mathematical structures when it is convenient and ignored them otherwise. In short, systems of mathematical structures have not really been used as models to guide research or as models to describe the thought processes of children. Instead, mathematical systems have usually been used simply as a basis to argue by analogy.

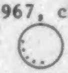
Martin (1976c) has explored some implications of using the Erlanger Programm (or some other "topological to projective to Euclidean" organization of geometries) as a model for research. In the remainder of this article, I would like to explore some similarities between the evolution of isometries (i.e., the rigid motions: slides, flips, and turns) and the evolution of spatial concepts in general. Then, the "topological to projective to Euclidean" issue will be reconsidered.

Isometries and Projections

"To understand a system of transformations" means that a child will conceive of states as end-products of transformations--and eventually transformations will be viewed as modifications of states. Initially, however, end-states tend to be viewed in isolation, as though they were unconnected to other states. For example, the following "perspective" task illustrates how young children repeatedly demonstrate an inability to take the point of view of another person, or to conceive of their own point of view as one of many possible points of view.

If a kindergartener is shown three different colored blocks on a table ( ), and if he is shown pictures ( ) depicting various points of view of the blocks, he can often select the picture that shows what he would see at various positions around the table. Nonetheless, if a doll is seated at some position different from the child's, kindergarteners are typically unable to select the picture of what the doll sees. After obvious consideration of the doll's point of view, kindergarteners commonly reselect the picture showing their own point of view (Piaget & Inhelder, 1967, chap. 8).

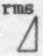
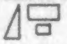
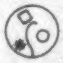
Piaget has explained that a child is not aware that he has a particular point of view until he realizes there are others, and he is aware of this only when he has coordinated all points of view into one operationally connected structure. To discover his own viewpoint, the child must relate it to others, distinguish it from the others, and coordinate it with the others. Until this is the case, the child's reasoning tends to resemble unconnected "snapshots" of beginning points or endpoints of transformations or operations. Further, he fails to notice information that becomes significant due to its invariance under a system of transformations. To illustrate this point, suppose a photograph of an apparent straight tree () is given to an adult, and the task is to determine whether the tree was, in fact, straight. By referring to the information in the photo, the question is unanswerable. From some other point of view, the tree might look curved ( or ). Straightness attains significance in the above snapshots only if isolated photographs are coordinated with the other points of view. Straightness is only read out of objects after certain information is read in either by measuring (a Euclidean concept) or by viewing the object from various points of view (a projective concept), or by applying some other system of operations.

To see how the above fact is reflected in the performance of children, a kindergartener can be shown a straight line on a round table ( ). If he is given small telephone poles and is asked to put them along the straight road, he will usually be able to do so. However, if the road is not present and only two endpoints are in place ( ), the response is often not a straight line at all (Piaget & Inhelder, 1967, chap. 6). Rather, the poles tend to follow the curve of the table (). Further, after placing his telephone poles in a curve, the child may recognize the line he has constructed is not straight, but be unable to improve his response. With respect to the development of mathematical-geometric concepts, what a child perceives is much less important than the rules of organization he gradually organizes to control and use the information he receives.

An important point to notice about the above tasks is that moving around a table is equivalent to remaining stationary and rotating the table. Consequently, Piaget's "perspective" tasks are closely related to tasks involving slides, flips, and turns (Piaget & Inhelder, 1967, p. 190). Piaget's "topological" concepts are those involving properties (e.g., touching, next to, etc.) that are important within a particular point of view--within a particular fixed state--independent of connections with other states. Piaget's "projective" concepts are those involving properties which become important when various points of view are connected by a system of transformations. Piaget's "Euclidean" concepts are those in which the observer becomes one of the transformed objects; that is, all objects are located with respect to fixed points of reference. Piaget makes the distinction between topological and Euclidean space as follows.

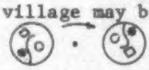
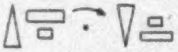
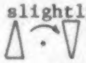
> Psychologically speaking, we may say that space becomes Euclidean when topological space is structured by reference elements, since the use of such elements brings about the distinction between two kinds of spatial reality, these being fixed 'sites' and space taken up by movable objects. In topological space, no distinction is drawn between container (fixed 'sites') and contained (movable objects), but in Euclidean space that distinction is constantly to the fore. (Piaget et al., 1960, p. 392)

Piaget's claim that spatial concepts develop from "topological to projective and Euclidean" is based on the assumption that the first spatial relations children use are those involving the simplest systems of transformations, that is, those not involving coordination of various points of view or the coordination of all objects (including the observer) within a single frame of reference.

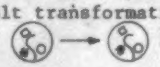
One of the cornerstones of Piagetian theory rests on the psychological viability of analyzing and equating tasks (and concepts) on the basis of their underlying operational structures. Yet, transformation tasks which are operationally isomorphic often vary widely (e.g., the equivalent of several years) in degree of difficulty. Consequently, if operationally isomorphic tasks differ "too much," it may be meaningless to equate tasks on the basis of operational structure. For instance, examples will be given in the following three sections to illustrate some of the characteristics that contribute to the difficulty of translations, rotations, and reflections. The objective will be to discuss whether it is sensible to characterize transformation geometry tasks in terms of slides, flips, and turns. In the examples that are given: (a)  represents simple geometric figures that can be moved on a desk top; (b)  represents configurations of simple figures that can be moved on a desk top; (c)  represents a model village layout that can be moved from one table to another; (d) represents chairs (or large flat shapes like those used on desk tops) that can be moved around a room. In each case, the tasks are posed by showing an initial figure (or configuration of figures) and by describing a transformation. The task is to construct (or describe or select from predrawn drawings) the final figure.

The examples will show that before a child has coordinated a particular system of transformations, he will (a) focus on only the most obvious (to the child) features of the transformed figure, ignoring other properties, (b) fail to notice information that only attains significance by being invariant under the system of transformations, and (c) distort the information he receives due to the influence of concepts that do not depend upon the system of transformations.

The Complexity of the Figure Transformed

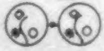
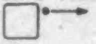
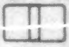
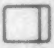
A complex village may be more difficult to rotate than a simple village (e.g., . However, transformations involving configurations of two or three figures (e.g., ) are often only slightly more difficult than transformations on simple figures (e.g., ). This fact seems to indicate that the difficulties involved in orienting a single geometric figure are very much like the difficulties involved in orienting simple configurations of figures. Nonetheless, if the configuration becomes too complex (as in the village layout), the operational complexity of the task increases. That is, the task is no longer just a transformation task; it is also a task testing a child's ability to notice multiple properties of the original figure.

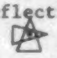
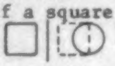
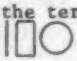
The more complex a figure becomes, the more active a child must be to attend to all of the relevant information that is available. For instance, if a kindergarten child is shown the following array of counters ( ) the amount of information he will be able to extract will be influenced by his level of mastery of certain ordering relations and classification operations (Lesh, Elwood, & Hall, 1976). That is, he could notice that the counters are in rows and columns; that the number of circles in each column is one, two, and three; that the overall shape is triangular or like a stair step, etc. If the child is not "operational" with respect to elementary seriation and classification operations, he will tend to center on only the most salient features of the dot pattern, ignoring other properties. Consequently, if the child is asked to perform a transformation on array counters, the task will be more difficult than if he is simply asked to recognize a correct transformation. Similarly, a configuration that a child has constructed will be easier to transform than a configuration he has watched someone else construct, and this, in turn, is easier than a configuration the child has been given without watching the initial construction (Piaget & Inhelder, 1971).

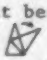
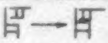
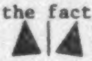
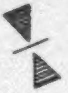
Another interesting phenomenon that occurs when complex figures are transformed is that properties preserved under simple transformations may not be preserved under more difficult transformations. For example, children's responses to a translation () may be classified as progressively emphasizing "topological," "projective," and then "Euclidean" properties (Piaget & Inhelder, 1967). However, a child who preserves "Euclidean" properties on the translation task may regress to focusing on "topological" properties on a rotation task. So this "topological projective to Euclidean" phenomenon is not as clear cut as one would hope. Nonetheless, it does seem to be true that properties preserved under simple transformations may be sacrificed under difficult transformations, and that the stages through which children progress (i.e., topological to Euclidean) are roughly the same for most tasks. However, even this statement will require modification. Some of the intricacies involved in the phenomenon are currently being investigated by Karen Schultz at Northwestern University.

Two types of operations are involved in transformation tasks: (a) operations and relations that are needed to extract information from the fixed states and (b) operations that are needed to organize the systems of transformations. Furthermore, just as preoperational children will tend to center on only the most obvious aspects of the fixed states, children who have not yet organized a system of transformations will focus only on the most obvious features of the motions involved. For instance, if rotations and reflections are taught using examples that have red flip lines and red centers of rotation, youngsters often fail to recognize simple transformations where the red dots and lines have been added inappropriately. That is, for the children, the red dots and lines become parts of the transformations (Lesh & Johnson, 1976).

The Size of the Transformation

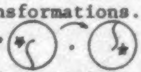
A "large" transformation (e.g.,) may be more difficult than a small transformation (e.g., ), but there are other confounding factors. For instance, if the transformation is too small, the initial and terminal states of the figure may overlap introducing new difficulties. For example, if a transparency covering a six inch square is slid four inches (e.g., ), and if a child is asked to draw what the figure will look like after the transformation, many children are reluctant to draw a square overlapping the original figure (e.g., ). A common response reported by Piaget and Inhelder (1971) is . Evidently children are much more willing to sacrifice squareness than they are "wholeness" (i.e., the fact that the initial figure was not divided by any lines).

Rotation and reflection tasks in which the fixed points are internal to the figure (e.g., ) also involve "pseudo-conservation" problems (i.e., distorting invariant properties that are not understood in order to preserve properties that are understood but which are not invariant). In fact, children sometimes refuse to represent what they have seen in order to represent what they understand. For instance, if a square (with a transparent interior) is flipped onto a circle (e.g., ) children are often reluctant to draw the circle inside (or on) the square in the terminal configuration. For example, one type of incorrect response is . The properties children tend to pseudo-conserve are often those Piaget has classified as "topological" (e.g., inside, on, outside) rather than properties like shape and size.

Sometimes small rotations (e.g., rotations of 30°) are simpler than large rotations (e.g., rotations of 180° or rotations around centers that are far from the original figure). But, small rotations can also be difficult because small rotations tend to involve overlapping figures (e.g., ). On the other hand, transformations that are "too large" can also involve confounding factors. For instance, if a translation is performed on figures in a large room (e.g., a gym), it is sometimes difficult for a child to establish frames of reference to orient the terminal configuration. That is, if several large geometric figures are slid fifteen feet, it may be difficult for a child to correctly orient the terminal configuration (e.g., ). Nontheless, the child may be able to perform relatively large translations on a desk top where it is easier to compare the initial and terminal configurations within one frame of reference (i.e., both are oriented relative to the child). For desk top transformations, external frames of reference are almost built into the task and need not be established by the child. However, even in the case of transformations in a large room, the influence of external frames of reference is evident from the fact that environmentally oriented figures (e.g., a chair in an upright position) are usually somewhat easier to transform than environmentally disoriented figures, or from the fact that horizontal (or vertical) slides, flips, or turns (e.g., ) are easier than oblique transformation (e.g., ).

The Number of Compositions

A single transformation is usually slightly easier than a composition of two transformations (e.g., a flip followed by a flip). However, recent studies at Northwestern indicate that the difference in difficulty is often surprisingly small. This suggests that the difficulties involved in a single transformation may be the same as those involved in compositions of transformations. An explanation of this rather surprising phenomenon can be found in Piaget's hypothesis that mathematical operations (or relations or transformations) do not exist in isolation. According to Piaget's wholism concept, to master a transformation implies that a simple system has been mastered; the simplest system (i.e., a grouping) involves (a) the transformation, (b) its inverse, (c) the identity transformation, and (d) compositions of pairs of these three types of transformations.

A phenomenon even more startling than the one mentioned above is that children are sometimes more confused about the terminal configuration of a single transformation than they are about the terminal configuration of a composition of several transformations. For example, if a simple village layout is rotated (e.g., ), and if the task is to place "x" marker in the appropriate position on the transformed figure, the task is often easier after a composition of four or five transformations (see Figure 19) than after a single transformation. The phenomenon seems to occur because the child ceases to think about the continuous motion connecting the end states and seemingly reminds himself, "The end state is just like the initial state." In short, the complexity of the composed transformation forces the child to behave like a sensible adult. Nonetheless, if "too many" transformations are composed, new difficulties are again introduced. For instance, if the transformed figure is an equilateral triangle, whose sides are red, white, and blue, respectively, and if the "x" is in the middle of the blue side, the child may not be convinced that the "x" will always be in the blue side. After 100 flips he may believe the "x" could be in the middle of the red side or the white side (Piaget & Inhelder, 1971). Arbitrarily large numbers of operations are confusing ideas for youngsters.

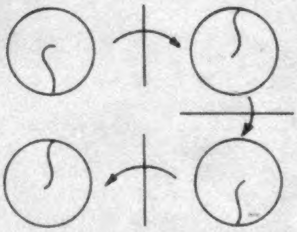
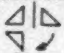
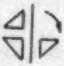


Figure 19. A composition of three turns.

A final factor that effects the difficulty of compositions is the order in which the individual transformations are composed. For instance, a flip followed by a turn (e.g., ) may involve different difficulties than a turn followed by a flip (e.g., ), even though the terminal states are identical for each of the compositions.

Slides to Flips to Turns

It is well known that any translation (slide) is equivalent to a composition of two rotations (turns), and that any rotation is equivalent to a composition of two reflections (flips). Consequently, two figures that are equivalent under slides are automatically equivalent under turns and flips. So, in a certain sense, slides are included in turns, which are in turn included in flips.

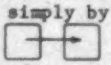
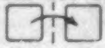
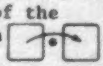
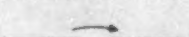
Figure 20. The inclusive relationship among slides, turns, and flips.

In the same way that Figure 16 gave rise to the conjecture that geometric concepts develop from topological to projective to Euclidean, Figure 20 might lead one to naively hypothesize that isometries are mastered in the order flips, turns, and slides. However, there are three issues which may make this conjecture untenable.

1. Just because a relation is mathematically the most general (or powerful) does not necessarily mean that it will be psychologically the most basic. For instance, most adults would probably predict that isometries are mastered in the order slides, turns, and flips (or slides, flips, and turns), and the only way to resolve the issue is to do research with children. Nonetheless, mathematical descriptions of isometries could be very helpful to use in directing research efforts.

2. Children often make mathematical judgements using qualitatively different systems of relations than those used by adults. Consequently, the researcher who begins with the assumption that children think in terms of slides, flips, and turns may be just as naive as the theorist who assumes flips come before turns and slides, just because flips are mathematically the most powerful. It could be that children do not conceive of rigid motions as compositions of slides, flips, and turns, but instead use some entirely different system of relations to describe spatial transformations. This is why it is important to occasionally use indirect research techniques. Otherwise, it is very easy to impose inappropriate mathematical structures on the thought processes of children.

3. If operationally isomorphic tasks vary too much in difficulty, it may be meaningless to equate tasks on the basis of operational structure. Research currently underway at Northwestern indicates that translations are generally easier than rotations, and reflections may be slightly easier than some rotations. But, it is quite easy to devise tasks where translations are more difficult than many reflections or rotations, or where rotations and translations are nearly equivalent in difficulty. Consequently, because some rotation tasks seem more closely related to reflection tasks than to other rotation tasks, the policy of classifying rigid motions into "slides," "flips," "turns" seems questionable. Perhaps there is a better basis for classifying transformation tasks.

Without changing the initial and final states of transformation, it is possible to alter a task simply by varying the description of the transformation (e.g., slide  , flip  , turn ). For instance, if a stick is moved from one position to another (e.g., ), children's drawings of the final state will vary in length depending on whether the stick was described as having been slid, flipped, or turned to the final position. Nonetheless, the appropriate way to analyze the tasks may not be to focus on slides, flips, and turns. For example, the important properties may be related to "up-down" and "left-right" position changes (Moyer, 1974).

Topological to Projective to Euclidean

In many respects the question of whether geometric concepts develop from topological to Euclidean is similar to the question of whether children's "rigid motion" concepts develop from slides to flips to turns. In particular, the same three issues that were discussed in the previous section can be considered concerning the development of geometric concepts in general.

One of the most important cornerstones of Piagetian theory rests on the psychological viability of analyzing, ordering, and equating tasks (and concepts) on the basis of their underlying operational structure. Consequently, the biggest threat to Piaget's point of view is not the fact that some "Euclidean" concepts are easier than some "topological" and "projective" concepts, or that children sometimes do not use Euclidean relations to make judgements about concepts that adults consider to be Euclidean or that the operational development of children can be accelerated or retarded by various educational or cultural factors. From the point of view of the present article the most important challenge to Piagetian theory is that operationally isomorphic tasks often vary so much in difficulty (especially in upper elementary school) that it may be meaningless to classify concepts on the basis of operational structure. Or, perhaps we have been investigating the wrong question?

Summary

The following questions have been raised in this paper.

1. If it is true that children's first spatial concepts involve "topological" relationships and if upper elementary school children's concepts begin to be characterized by formal operational structures (e.g., INRC groups), what inferences can be made about the sequential evolution of concepts at intermediate stages of development? Perhaps known mathematical structures can be used to investigate these types of questions.

2. What sorts of qualitative differences may exist between children's and adults' conceptions of various matehmatical concepts? Perhaps indirect research techniques could provide valuable information concerning these types of questions.

3. What factors influence the difficulty of tasks that are characterized by isomorphic structures? Piaget (Laurendeau & Pinard, 1970) has described several factors that account for décalages involving operationally isomorphic tasks that differ only in figurative content. However, Piaget generally tended to de-emphasize the importance of décalages, prefering to focus on the analysis of ideas rather than on analysis of concrete materials and figurative models.

In the introduction to Laurendeau and Pinard's "Space" book (Laurendeau & Pinard, 1970), Piaget stated:

> Décalages derive from the object's resistances, and the authors [Laurendeau and Pinard] ask that we construct a theory of these resistances, as though this were an undertaking directly parallel to the one concerning the subject's actions and operations. It is an exciting project and it should certainly be considered. But we must remark at once that if the subject's actions always reflect intelligence (a condition which greatly facilitates the analysis), the object's resistances do not do so, and involve a much greater number of factors. (p. 4)

Nonetheless, Laurendeau and Pinard's charge seems justified that Piaget makes too frequent use of décalages. For example, it is ironic that Piaget, who has done as much as anyone to clarify the nature of concepts like mass, weight, and volume, still continues to speak of "décalages" between conservation of mass, conservation of weight, and conservation of volume--as though such tasks were operationally isomorphic. No wonder so many psychologists have made the error of attempting to describe the acquisition of "conservation" as though conservation of what (i.e., mass, weight, or volume) were unimportant.

Some of the most important goals of space research are: (a) to thoroughly investigate the operational structure of a variety of geometric concepts, (b) to analyze relationships between various geometric concepts, and (c) to clarify some of the relations between figurative and operative aspects of thought. Even if psychologists believe they can ignore variations due to figurative content, the issue is highly important to educators who must use concrete materials and figurative models to teach mathematical concepts.

The goal of this article has been to use transformation geometry as a context in which to discuss relationships between mathematical structures, cognitive structures, and instructional structures. By examining justifications for teaching geometry in elementary school (i.e., instructional structures), some concepts (i.e., mathematical structures) were mentioned that psychologists have tended to neglect. To investigate the psychological status (i.e., cognitive structure) of these mathematical ideas, mathematical structures can be used to guide indirect research procedures. When these procedures are used, it becomes clear that children sometimes make mathematical judgements using qualitatively different systems of relations than those typically used by adults. That is, even though there are striking similarities between mathematical structures and cognitive structures, there are also some important dissimilarities. These dissimilarities sometimes make it difficult to interpret the mathematical status of concepts psychologists have studied. Nonetheless, if information about children's cognitive structures is ever to be useful in organizing instructional activities for children, it is likely that known mathematical systems will have to be used as a guide.

Transformation geometry seems to furnish an excellent guide in which to investigate (a) the extent to which known mathematical systems can be used to model the sequential development of children's mathematical concepts, (b) the psychological viability of analyzing, ordering, and equating tasks on the basis of their underlying operational structure, and (c) relationships between figurative and operative aspects of thinking.

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